KNOT THEORY

lent term

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0] Foundations

0.1 Isotopies

Suppose $f: M \to N$ is a map of smooth manifolds.

Dfn: f is an embedding $M \hookrightarrow N$ if $df: TM \to TN$ is injective

By the Inverse Function Theorem : if $df|_x$ is injective, then $\exists$ U nbhd of $x \in M$ such that $f|_U$ is an embedding.

> Remember locally that $DF: TM \to TN$ (push forward by F) maps $F_*: \frac{\partial}{\partial x_i} \mapsto \sum_j \frac{\partial F^j}{\partial x^i} \frac{\partial}{\partial y^j}$. The map $df|_x$ is then a map $df|_x: T_x M \to T_{f(x)} N$, which is an injective linear map
>
> Embedding = immersion that's homeo onto it's image. We have this lemma from diff geo:
>
> Lemma 5.2: If D_pF is an isomorphism, then $\exists$ open neighbourhoods U of p, V of $F(p)$ such that $F|_U: U \to V$ is a diffeomorphism.
>
> Proof: pick charts φ about p, ψ about $F(p)$. Then $g := \psi \circ F \circ \varphi^{-1}$ is a map $\mathbb{R}^n \to \mathbb{R}^m$ with invertible derivative at $\varphi(p)$. By inverse function theorem, there exist open neighbourhoods U' of $\varphi(p)$, V' of $\psi \circ F(p)$ such that g is a diffeomorphism $U' \to V'$. But this says precisely that g is a diffeomorphism $g: U \to V$, where $U := \varphi^{-1}(U)$, $V := \psi^{-1}(V')$. $\square$

Dfn: if $f_0, f_1: M \hookrightarrow N$ are embeddings, f_0 is isotopic to f_1 ($f_0 \sim_i f_1$) if there is a smooth map $F: M \times I \to N$ with $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$ and $f_t(x) = F(x,t)$ is an embedding $\forall t$

Clearly isotopy $\Rightarrow$ homotopy. We call F an isotopy.

Lemma: if $f_0 \sim_i f_1$ via F, then $f_0 \sim f_1$ via $\hat{F}$ with $\hat{F}(x,t) = f_0(x)$ $t \in [0, 1/4)$, $\hat{F}(x,t) = f_1(x)$ with $t \in (3/4, 1]$

proof: choose $g: I \to I$ smooth with $g(t) = 0$ $t \in [0, 1/4)$, $g(t) = 1$ for $t \in (3/4, 1]$

Then take $\hat{F}(x,t) = F(x, g(t))$

Corollary: isotopy is an equivalence relation.

> Reflexive: Constant isotopy $H: f_0 \sim f_0$, $H(x,t) = f_0(x)$. Symmetric: Take $H(x, 1-t)$
>
> Transitive: suppose $F: f_0 \sim f_1$ and $G: f_1 \sim f_2$. Then using the above idea, let $\hat{F}: f_0 \sim f_1$ be an isotopy from F s.t $F(x,t) = f_1(x)$ for $(1/4, 1]$, and $\hat{G}: f_1 \sim f_2$ be the isotopy from G s.t $\hat{G}(x,t) = f_1(x)$ on $[0, 3/4)$
>
> Then $\hat{F}$ and $\hat{G}$ agree on $(1/4, 3/4)$, and in particular $D\hat{F} = D\hat{G}$ on $(1/3, 2/3)$. So we can write down a smooth map $H: M \times I \to N$:
>
> $$H(x,t) = \begin{cases} \hat{F}(x,t) & t \in [0, 1/3) \\ f_1(x) & t \in [1/3, 2/3] \\ \hat{G}(x,t) & t \in (1/3, 1] \end{cases}$$
>
> which satisfies (1) $\forall t \in [0,1]$, $H_t(x)$ is an embedding and (2) $H_0(x) = f_0(x)$ and $H_1(x) = f_2(x)$. So H is an isotopy $f_0 \sim f_2$. Thus $\sim$ is an equivalence relation. $\square$

Ex 1: if $\vec{v}(t)$ is a smooth, compactly supported, time dependent vector field on M, then there's an isotopy (the flow of $\vec{v}$) $\Phi : M \times [0,1] \to M$ with $\Phi_0 = id_M$ and $\left.\frac{d\Phi}{dt}\right|_{(x,t)} = v(x,t)$

> This is a nice result from symplectic geometry.

Ex 2: if $f : \mathbb{R}^m \to \mathbb{R}^n$ with $df|_0$ injective, then $\exists\ U$ nbhd of $0 \in \mathbb{R}^m$ s.t $f|_U \sim_i df|_0|_U$

identify tangent space with $\mathbb{R}^n$. or $\mathbb{R}^m$?

proof: $F(x,t) = t f(x) + (1-t)\, df|_0(x)$

Then $dF_t|_0 = t\, df|_0 + (1-t)\, df|_0 = df|_0 \Rightarrow \exists\ U_t$ s.t $F_t|_{U_t}$ is an embedding.

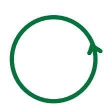
To get a uniform U, consider $dF|_{(0,t)} = df|_0 \oplus id$ injective. We can find an ε s.t $F|_{B_\varepsilon(0) \times [t-\varepsilon,\, t+\varepsilon]}$ is an embedding using compactness.

> Thoughts: The map $df|_0$ acts on $T_0\mathbb{R}^m \to T_{f(0)}\mathbb{R}^n$, which we can canonically identify with $\mathbb{R}^m$ and $\mathbb{R}^n$. That is, we consider $df|_0 : \mathbb{R}^m \to \mathbb{R}^n$.
>
> So define $F : \mathbb{R}^m \times I \to \mathbb{R}^n$; $(x,t) \mapsto t f(x) + (1-t)\, df|_0(x)$. I'm not 100% convinced why, but letting $F_t(x) := F(x,t)$, we have
>
> $$\begin{aligned} dF_t|_0 &= d\left(t f(x) + (1-t)\, df|_0(x)\right)\big|_0 \\ &= t\, df|_0(x) + (1-t)\, df|_0(x) \quad ? \\ &= df|_0 \end{aligned}$$
>
> By assumption, $df|_0$ is injective, so $dF_t|_0$ is injective. By our very first remark then, $\exists$ nhood U of 0 in $\mathbb{R}^m$ s.t $F_t|_U$ is an embedding. Now we want to find a U that is uniform (works) for all t. Note that $dF|_{(0,t)} = df|_0 \oplus id$ (taking derivative in each component ?), which is also injective. So by the same argument, we can find an open nhood U' of $(0,t)$ s.t $F|_{U'}$ is an embedding. By compactness of $\mathbb{R}^m \times I$, we can wlog take $U' = B_\varepsilon(0) \times [t-\varepsilon, t+\varepsilon]$ for some $\varepsilon > 0$.
>
> So what does this all say? Well, $F|_{U'}$ is a smooth map $\mathbb{R}^m \times I|_{U'} \to \mathbb{R}^n$, such that
>
> (1) $(F|_{U'})_t$ is an embedding $\forall t$ $\quad ((F|_{U'})_t = F_t|_V)$
>
> (2) $(F|_{U'})_0 = df|_0|_V$
>
> (3) $(F|_{U'})_1 = f|_V$
>
> So that actually $F|_{U'} : f|_V \sim df|_V$
>
> Some things I'm not too sure about here, like this derivative looks sus.

0.2 Knots

Dfn: an **oriented knot** in $\mathbb{R}^3$ is an isotopy class of embeddings $K : S^1 \hookrightarrow \mathbb{R}^3$.

Example: the unknot is the class of $U : S^1 \to \mathbb{R}^3$; $(x,y) \mapsto (x,y,0)$



$df|_0(x)$

Exercise: if $\varphi: S^1 \to S^1$ is an orientation-preserving diffeomorphism, then $\varphi \sim_i$ identity.

Hence: $\Rightarrow$ $K \circ \varphi \sim_i K \circ id_{S^1} = K \Rightarrow$ reparametrizing w/o changing isotopy class

The general idea comes from the fact that $Diff(S^1)$ has two connected components: orientation-preserving and orientation-reversing. Of course, $id: S^1 \to S^1$ is orientation-preserving. How do we formalize this? Consider that S^1 can be thought of as $\mathbb{R}/\mathbb{Z}$, and in particular any diffeo $\varphi: S^1 \to S^1$ can be lifted to a diffeo $\tilde{\varphi}: \mathbb{R} \to \mathbb{R}$ satisfying (1) $\tilde{\varphi}(0) \in [0,1)$ and $\tilde{\varphi}(x+1) = \tilde{\varphi}(x) + 1$. Moreover, this lift is unique (obviously completely determined by φ on $[0,1]$). Also, id_{S^1} lifts to $id_{\mathbb{R}}$. To show $\varphi \sim_i id_{S^1}$, it suffices to show that $\tilde{\varphi} \sim_i id_{\mathbb{R}}$, and then the result will follow by taking the quotient.

But of course, since we are in $\mathbb{R} \to \mathbb{R}$, we can just take the path isotopy:

Could perhaps say $\tilde{\varphi}(x) = \varphi(x)$ on $[0,1)$ instead of $\tilde{\varphi}(0) \in [0,1)$.

$H: \mathbb{R} \times I \to \mathbb{R}; \quad (x,t) \mapsto t\tilde{\varphi}(x) + (1-t)x$

This is clearly a smooth map, and satisfies

(1) $H_0(x) = id_{\mathbb{R}}$ (2) $H_1(x) = \tilde{\varphi}(x)$

this follows because its orientation preserving I believe.

(3) For each t, $H_t(x)$ is an embedding. In particular, it satisfies

$$H_t(0) = t\tilde{\varphi}(0) + (1-t)0 \in [0,1)$$

$$H_t(x+1) = t\tilde{\varphi}(x+1) + (1-t)(x+1) = t\tilde{\varphi}(x) + (1-t)x + t - t + 1 = H_t(x) + 1$$

$\Rightarrow$ H_t descends to a diffeo $h_t: S^1 \to S^1$, $h_0 = \varphi$ and $h_1 = id_{S^1}$. So $h: \varphi \sim_i id$. $\square$

If $\varphi \sim_i \phi$, then for any diffeo $\gamma: S^1 \to S^1$, $\gamma \circ \varphi \sim_i \gamma \circ \phi$ and $\varphi \circ \gamma \sim_i \phi \circ \gamma$.
This follows since the composition of an embedding and a diffeo gives an embedding, and hence composition of an isometry and a diffeomorphism gives an isometry. ?

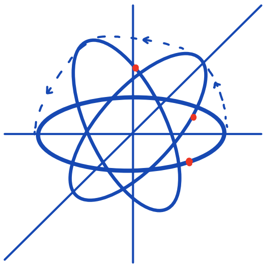
Dfn: the reverse of K is $r(K) = K \circ r$ where $r: S^1 \to S^1$; $(x,y) \mapsto (x,-y)$

Exercise: $U \sim_i r(U)$.

The idea is to rotate the unknot π radians about the x-axis, as seen in the diagram

$U: S^1 \hookrightarrow \mathbb{R}^3; \quad (x,y) \mapsto (x,y,0)$

$r(U): S^1 \hookrightarrow \mathbb{R}^3; \quad (x,y) \mapsto (x,-y,0)$



Write down: $H: S^1 \times I \to \mathbb{R}^3; \quad ((x,y),t) \mapsto (x, y\cos(\pi t), y\sin(\pi t))$. Then observe

(1) H is a smooth map

(2) $H_0(x,y) = (x, y\cos(0), y\sin(0)) = (x,y,0) = U$

$H_1(x,y) = (x, y\cos(\pi), y\sin(\pi)) = (x,-y,0) = r(U)$

(3) for any fixed $t \in I$

$$H_t(x,y) = (x, y\cos(\pi t), y\sin(\pi t))$$

is an embedding of the circle S^1 into $\mathbb{R}^3$ $\square$

Dfn: $\{\text{knots in } \mathbb{R}^3\} = \{\text{oriented knots in } \mathbb{R}^3\}/\sim$ where we identify $K \sim r(K)$.

0.3 Knot diagrams

Dfn : A knot diagram is

a) a smooth map $\gamma : S^1 \to \mathbb{R}^2$ s.t

1. $\gamma'(p) \neq 0 \ \forall p \in S^1$ (so ≺ is not a knot diagram)

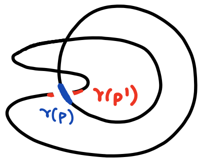
2. if $\gamma(p) = \gamma(p')$, then $\gamma(p)$ and $\gamma(p')$ are linearly independent (transverse double points)

no tangent double points

3. $\nexists$ distinct p, q, r with $\gamma(p) = \gamma(q) = \gamma(r)$ i.e. no triple points

b) an ordering $p > p'$ on each pair of double points $(\gamma(p) = \gamma(p'))$

e.g.



represents $p' < p$.

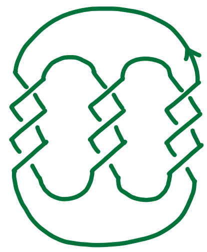
Choose $z : S^1 \to \mathbb{R}$ with $z(p) > z(p')$

whenever $\gamma(p) = \gamma(p')$ and $p > p'$.

Then define $k : S^1 \to \mathbb{R}^3$ by $k(p) = (\gamma(p), z(p))$. Choice of z doesn't matter, if $\hat{z}$ is another choice then $k \sim \hat{k}$ via $F(p,t) = (\gamma(p), t z(p) + (1-t)\hat{z}(p))$

Example :

unknot unknot unknot negative trefoil.



$K \neq r(K)$

If $\vec{v} \in S^2$, $\pi_v : \mathbb{R}^3 \to \mathbb{R}^2$ is orthogonal projection.

Theorem: Given $k : S^1 \hookrightarrow \mathbb{R}^3$, there is an open dense subset $U \subset S^2$. such that $\pi_v \circ k$ is a knot diagram $\forall \vec{v} \in U$ ($p > p'$ if $v \cdot k(p) > v \cdot k(p')$). (dot product)

Essentially take any knot $K : S^1 \hookrightarrow \mathbb{R}^3$. For almost all $\vec{v} \in S^2$, we can project K down onto the plane orthogonal to $\vec{v}$ via $\pi_{\vec{v}}$, and the resulting map $\pi_v \circ K$ will be a knot diagram

proof: Denote $\gamma_v = \pi_v \circ k : S^1 \to \mathbb{R}^2$. The question is: is γ_v a knot diagram? γ_v is a knot diagram if

1) $\gamma'(p) \neq 0 \quad \forall p$ (γ_v is immersed)

2) double points are transverse

3) no triple points.

We need a result to prove this theorem?

Suppose $f : M \to N$ is a smooth map. We say $x \in M$ is a critical point of f if $df|_x$ is not surjective, and $y \in N$ is a critical value of f if $f^{-1}(y)$ contains a critical point. Otherwise it's a regular value.

Sard's theorem : the set of critical values of f has measure 0 in N. So if M is compact, the set of regular values of f is open and dense in N.

Consider $\varphi : S^1 \times S^1 \to S^2$; $(p,q) \mapsto \rho(K(p) - K(q))$ for $p \neq q$, where $\rho : \mathbb{R}^3 \setminus \{0\} \to S^2$; $x \mapsto \frac{x}{\|x\|}$

$(p,p) \mapsto \rho(K'(p))$

We'll say some preliminary stuff and then invoke Sard's theorem to say that this holds.

For condition 1, remark that $\gamma'_v(p) \neq 0 \iff \pi_v(K'(p)) \neq 0$ (Taking derivative commutes with projection, draw examples)
If $\pi_v(K'(p)) = 0$, then this $\iff K'(p) = \lambda v$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, $\iff \rho(K'(p)) = \pm v$. So $\pi_v(K'(p)) \neq 0$ is equivalent to $\rho(K'(p)) \neq \pm v$.

So the condition $\gamma'(p) \neq 0 \ \forall p$ is equivalent to $\varphi(p,p) \neq \pm v$ (can write as $\varphi(\Delta) \neq \pm v$, where Δ denotes the diagonal)

For condition 2, remark that (p,q) is a double point of γ_v (i.e. $\gamma_v(p) = \gamma_v(q)$)

$$\begin{aligned}
&\iff \pi_v(K(p)) - \pi_v(K(q)) = 0 \\
&\iff \pi_v(K(p) - K(q)) = 0 \\
&\iff K(p) - K(q) = \lambda v \text{ for some } \lambda \in \mathbb{R} \setminus \{0\} \\
&\iff \rho(K(p) - K(q)) = \pm v \\
&\iff \varphi(p,q) = \pm v
\end{aligned}$$

($\lambda \neq 0$ since K an embedding: $K(p) = K(q) \iff p = q$)

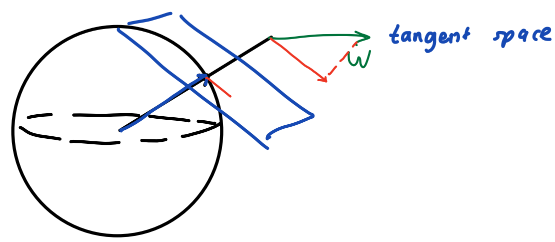
If (p,q) is in fact a double point of γ_v,

By chain rule:

$$\begin{aligned}
(d\varphi)_{(p,q)} &= \left(d\left(\rho(K(x) - K(y))\right)\right)\Big|_{(p,q)} \\
&= (d\rho)_{K(p)-K(q)}\left(d(K(x) - K(y))_{(p,q)}\right) \\
&= (d\rho)_{\lambda v}\left(K'(p)\,dx - K'(q)\,dy\right)
\end{aligned}$$

Chain rule:
$d_x(f \circ g) = d_{g(x)} f \circ d_x(g)$

So that $(d\varphi)_{(p,q)}(\alpha, \beta) = (d\rho)_{\lambda v}(\alpha K'(p) - \beta K'(q))$ for any tangent vector (α,β) at (p,q).



We can calculate that $d\rho_{\lambda v}(w) = \frac{1}{\lambda}\pi_v(w)$

So $\rho : \mathbb{R}^3 \setminus \{0\} \to S^2$; $x \mapsto \frac{x}{\|x\|}$. Now, $d\rho_{\lambda v}$ is then a map $T_{\lambda v}(\mathbb{R}^3 \setminus \{0\}) \to T_v S^2$

identify with $\mathbb{R}^3 \setminus \{0\}$

WLOG say $\lambda > 0$. The map $d\rho_{\lambda v}$ acts explicitly on the standard tangent vectors $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$

By

$$\begin{aligned}
d\rho_{\lambda v}\left(\frac{\partial}{\partial x^i}\right) &= \sum_j \left(\frac{\partial}{\partial x^i}\rho^j\right)_{\lambda v} \frac{\partial}{\partial y^i}\Big|_{\rho(\lambda v)} \\
&= \sum_j \left(\frac{\partial}{\partial x^i}\rho^j\right)_{\lambda v} \frac{\partial}{\partial y^i}\Big|_v
\end{aligned}$$

Want more heuristics.

which under the identification of $T_x\mathbb{R}^3 \cong \mathbb{R}^3$ and considering $S^2 \hookrightarrow \mathbb{R}^3$, This is the same as $\frac{1}{\lambda}\pi_v(e_i)$. I think the best way to see this is to do the computation explicitly in coordinates to see where the $\frac{1}{\lambda}$ term comes from. But it is clear that the above projects w onto the tangent space of S^2 at v, which we identify with the plane orthogonal to v. ?

Since $d\rho_{\lambda v}(w) = \frac{1}{\lambda}\pi_v(w)$, $(d\varphi)_{(p,q)}(\alpha,\beta) = (d\rho)_{\lambda v}(\alpha K'(p) - \beta K'(q))$

$$= \frac{1}{\lambda}\pi_v(\alpha K'(p) - \beta K'(q))$$
$$= \frac{1}{\lambda}\alpha\gamma_v'(p) - \frac{1}{\lambda}\beta\gamma_v'(q)$$

Hence $(d\varphi)_{(p,q)}$ is surjective iff $\gamma_v'(p)$ and $\gamma_v'(q)$ are linearly independent.

$(d\varphi)_{(p,q)}(\alpha,\beta)$ can be thought of as a map $\mathbb{R}^2 \to \mathbb{R}^2$ (identifying tangent spaces), and the above says that actually it is a linear map. By dimension reasons then, this map is surjective iff it's injective.

$$\underset{\substack{\uparrow \\ \text{domain}}}{\dim(\mathbb{R}^2)} = \underset{\substack{\| \\ \mathbb{R}^2}}{\dim(\mathrm{Im}(L))} \Leftrightarrow \underset{\substack{\| \\ 0}}{+\dim(\ker(L))}$$

But of course, $(d\varphi)_{(p,q)}$ is injective $\Leftrightarrow$ $(d\varphi)_{(p,q)}(\alpha,\beta) = 0 \Leftrightarrow (\alpha,\beta) = 0$.

But $\Leftrightarrow$ $\frac{1}{\lambda}\alpha\gamma_v'(p) - \frac{1}{\lambda}\beta\gamma_v'(q) = 0$

But the statement "$\frac{1}{\lambda}\alpha\gamma_v'(p) - \frac{1}{\lambda}\beta\gamma_v'(q) = 0 \Leftrightarrow (\alpha,\beta) = 0$" says exactly that $\gamma_v'(p)$ and $\gamma_v'(q)$ are linearly independent (getting rid of $1/\lambda$ coefficient).

With all this established, we can invoke Sard's theorem

Saard's theorem says that there is an open, dense set $U \subset S^2$ containing only regular values of φ, but specifically $U \cap \pm\varphi(\Delta) = \emptyset$. That says that

We want to say there's a dense subset of vectors $U \subset S^2$ such that (1) $\rho(K'(p)) \neq \pm v$ and (2) $\gamma_v'(p)$ and $\gamma_v'(q)$ are linearly independent when $\gamma_v(p) = \gamma_v(q)$ for $(p,q) \in S^1 \times S^1$. By Sard's theorem, we know $\exists$ an open, dense subset $U \subset S^2$ which contains only regular values: $v \in S^2$ s.t for any $(p,q) \in \varphi^{-1}(v)$, $d\varphi|_{(p,q)}$ is surjective. In particular, we can choose such a U s.t $U \cap \varphi(\Delta) = \emptyset$.

Notice that the double points of γ_v are exactly the preimage of v under φ: if $\gamma_v(p) = \gamma_v(q)$, then $K(p)$ and $K(q)$ differ in $\mathbb{R}^3$ by λv ($K(p) - K(q) = \lambda v$). Hence, $\varphi(p,q) = \rho(K(p) - K(q)) = \frac{\lambda v}{\|\lambda v\|} = v$. This argument is easily reversed.

Alright. So if $v \in S^2$ and p, q are double points of γ_v, then $(p,q) \in \varphi^{-1}(v)$. We showed above that $\gamma_v'(p)$ and $\gamma_v'(q)$ are linearly independent iff $(d\varphi)_{(p,q)}$ is surjective. Since we want this condition to hold $\forall$ double points, we want that $\forall (p,q) \in \varphi^{-1}(v)$, $(d\varphi)_{(p,q)}$ is surjective. This is equivalent to saying that v is a regular value of φ.

Therefore if $v \in S^2$, then condition 2 is only satisfied if v is a regular value of φ. Sard's theorem says that we can find an open, dense subset of such v.

probably to do with measure 0.

That deals with condition 2. For condition 1, remember we selectively chose U s.t $U \cap \varphi(\Delta) = \emptyset$ so certainly $\forall p \in S^1$, $\varphi(p,p) \neq \pm v$ for any regular value v of φ. So on U, condition 1 is automatically satisfied.

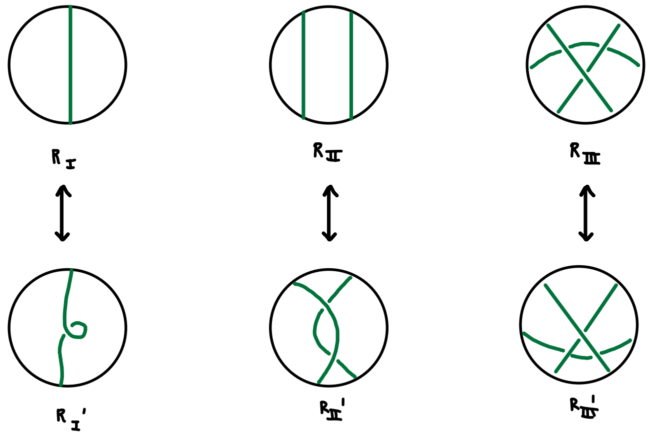
3: Similar: Show that if 1 and 2 hold at v, there's a nearby v' for which 3 holds. $\square$

need to still do.

0.4 Reidermeister Moves

Problem: given D and D' which represent the same knot K, how are D and D' related?

Reidermeister moves:



Dfn: diagrams D and D' are locally equivalent if there is $A \subseteq \mathbb{R}^2$ s.t $A \simeq D^2$ such that

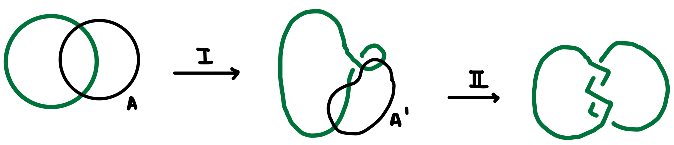
$$D \cap (\mathbb{R}^2 - A) = D' \cap (\mathbb{R}^2 - A)$$

and homeomorphisms $\varphi : (A, D \cap A) \xrightarrow{\sim} (D^2, R_i)$

$\varphi' : (A, D' \cap A) \rightarrow (D^2, R_i')$

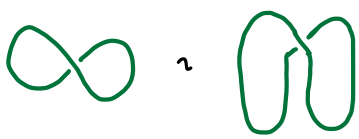
essentially you can perform Reidermeister moves locally to get from one to another

Example:



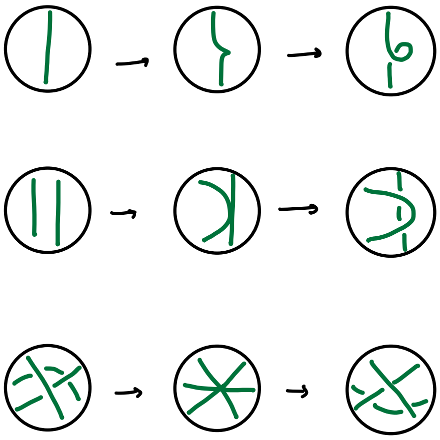
Theorem: Reidermeister let $\sim$ be the equivalence relation on diagrams generated by local moves and diffeomorphisms of $\mathbb{R}^2$

e.g.



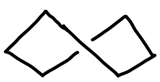
If D and D' represent isotopic knots K and K', then $D \sim D'$.

Proof of Reidermeister moves.



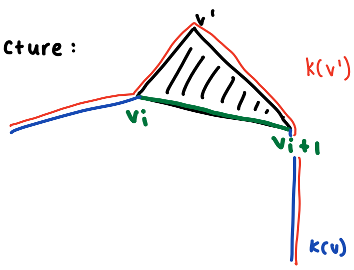
PL Knots : if $V = (v_0, \dots, v_n) \in (\mathbb{R}^3)^{n+1}$ with $v_0 = v_n$, let $K(V) = \bigcup_{i=1}^{n} \underbrace{\overline{v_{i-1} v_i}}_{\text{line segment}}$

Dfn: $K(V)$ is a PL Knot if $\overline{v_{i-1} v_i} \cap \overline{v_{j-1} v_j} = \emptyset$ for $i \neq j-1, j, j+1$.

e.g. 

Dfn: Suppose $K(V)$ is a PL knot, and $v' \in \mathbb{R}^3$ with the triangle $(\Delta v_i v' v_{i+1})$ having $(\Delta v_i v' v_{i+1}) \cap K(V) = \overline{v_i v_{i+1}}$. We say $K(V)$ is locally equivalent to $K(v_0 \dots v_i v' v_{i+1} \dots v_n) := K(V')$

Picture:



idea: $K(v') \sim K(V)$

(PL isotopic)

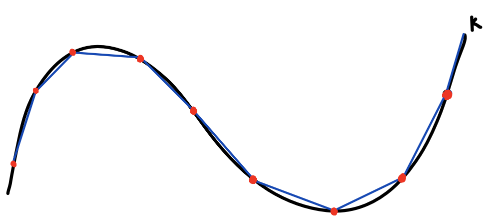
Dfn: PL equivalence is the equivalence relation on PL knots generated by local equivalence.

Thm: There's a bijection $\{\text{smooth knots}\} \longleftrightarrow \{\text{PL knots}\}/\text{isotopy}$

$$K \longleftrightarrow L(K)$$

if K is piecewise smooth isotopic $L(K)$

Picture:

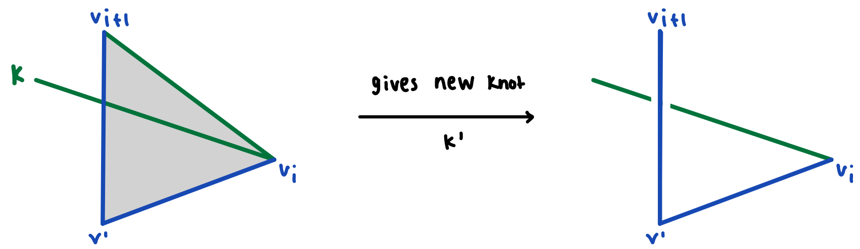


We can consider knots as PL knots, and also by using this triangle trick we can insert vertices or change our knot in some convenient way that gives us something that is equivalent to the original PL knot. We can use this then to show that the Reidermeister moves give the same knot using their PL analogues.

Suppose K and K' are locally equivalent. After subdividing triangles, assume that $\pi(\Delta v_i v' v_{i+1})$ intersects $\pi(K \setminus \overline{v_i v_{i+1}})$ in either

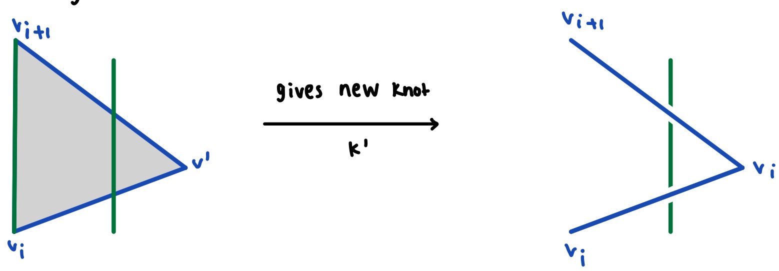
1) a line segment.

Note: green represents the knot, blue represents the triangle



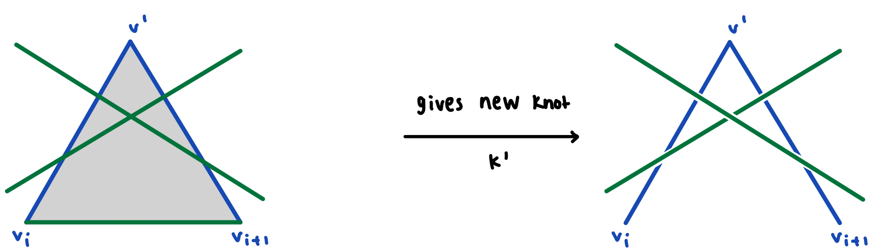
$\Rightarrow$ RI)

2) Two line segments



$\Rightarrow$ R II)

3) Two line segments with a single crossing

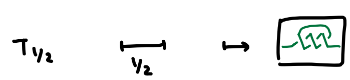


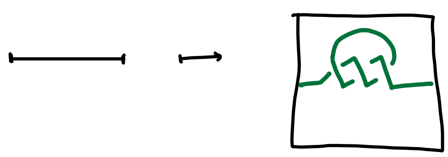
$\Rightarrow$ R III)

(under ~ over + crosses ~ over)

<u>Warning</u>: continuous maps are not your friends

e.g. $T : [-1,1] \to [-1,1]^3$

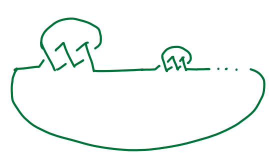




$f : I \to \mathbb{R}^3$

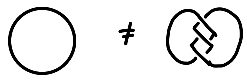
$T_s(t) : [-s,s] \to [-s,s]^3$

$T_s(t) = sT(t/s)$



1] Jones Polynomial

Motivation: want to show



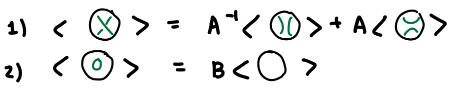
Idea : $I : \{\text{diagrams}\} \to S$, and if I doesn't change under Reidermeister moves, it descends to a map $I : \{\text{knots}\} \to S$

1.1) Kauffman Bracket

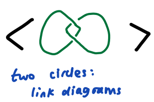
Prop: There is a unique map $\langle\ \rangle : \{\text{knot diagrams}\} \to \mathbb{Z}[A^{\pm 1}, B]$ satisfying

0) $\langle \emptyset \rangle = 1$

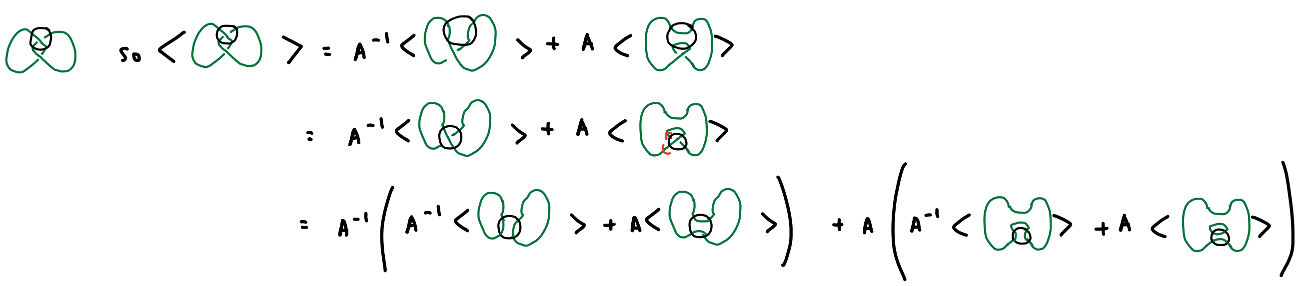
The local rules



Example :



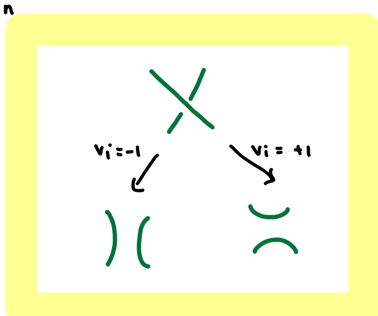
Pick :



$$= A^{-2}\langle \bigcirc \bigcirc \rangle + \langle \bigcirc \rangle + \langle \bigcirc \rangle + A^2 \langle \circ\ \circ \rangle$$

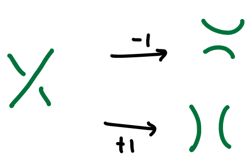
$$= A^{-2}B^2 + 2B + A^2B^2$$

proof of proposition: if D has n crossings, we can apply the rule to every one of them. The set of possible resolutions is in bijection with $\{\pm 1\}^n$



So given $v \in \{\pm 1\}^n$, assign D_v to it by resolving i^{th} crossing according to v_i as above.

Rem:

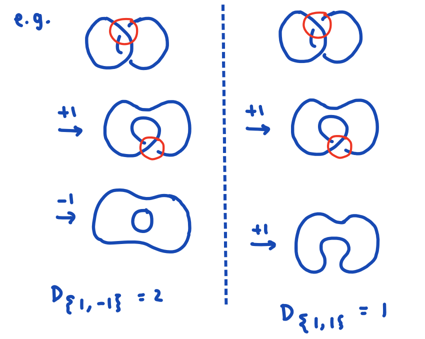


, reverse of above

Define $\langle D \rangle = \sum_{v \in \{\pm 1\}^n} A^{\sum v_i} B^{|D_v|}$, where $|D_v|$ = # of components in D_v.

D_v is the knot after resolving all crossings according to $v \in \{\pm 1\}^n$

e.g.



$D_{\{1,-1\}} = 2$ $\qquad$ $D_{\{1,1\}} = 1$

Prop : $\langle \ \ \rangle$ satisfies

1) $\langle \emptyset \rangle = 1$
2) $\langle \times \rangle = A^{-1} \langle)(\rangle + A \langle \asymp \rangle$
3) $\langle \circ \quad \rangle = B \langle \quad \rangle$

proof : (1) is obvious

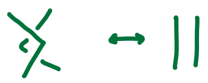
(2) $\langle \times \rangle = \sum_{\{v \mid v_j = -1\}} A^{\sum v_i} B^{|D_v|} + \sum_{\{v \mid v_j = +1\}} A^{\sum v_i} B^{|D_v|}$

j^{th} crossing

$= A^{-1} \langle)(\rangle + A \langle \asymp \rangle$

(3) if $D = \circ$, $D' =$, then $|D_v| = |D'_v| + 1$. □

Effect of R. Moves on $\langle \ \rangle$.



Note: orientation of crossing is very important to keep track of!

RII) $\langle \quad \rangle = A^{-1} \langle \quad \rangle + A \langle \quad \rangle$

$= \langle \,|\,|\, \rangle + A^{-2} \langle \asymp \rangle + A^{2} \langle \asymp \rangle + \langle \quad \rangle$

$= (A^{-2} + B + A^{2}) \langle \asymp \rangle + \langle \,|\,|\, \rangle$

From now on, take $\boxed{B = -A^2 - A^{-2}}$

$\Rightarrow \langle \quad \rangle = \langle \,||\, \rangle$

RIII) $\langle \quad \rangle = A \langle \quad \rangle + A^{-1} \langle \quad \rangle$ RII diagram

$= A \langle \quad \rangle + A^{-1} \langle \quad \rangle$

think about which way you view diagram.

$= A \langle \quad \rangle + A^{-1} \langle \quad \rangle$ by RII) invariance

$= \langle \quad \rangle$

So $\langle \ \rangle$ is also invariant under RIII.

RI) $\langle \quad \rangle = A^{-1} \langle \,|\circ\, \rangle + A \langle \quad \rangle$

$= (A^{-1}(-A^{-2} - A^{2}) + A) \langle \,|\, \rangle = -A^{-3} \langle \,|\, \rangle$

which is not invariant under RI.

If D is an oriented link diagram, then every crossing looks like

$-$ve crossing   +ve crossing

let $n_\pm(D) :=$ # of $\pm$ crossings. The writhe of D is $w(D) = n_+(D) - n_-(D)$.

Lemma: if D_i and D_i' are related by i^{th} Reidermeister move:

1) $w(D_1') = w(D_1) - 1$
2) $w(D_2') = w(D_2)$
3) $w(D_3') = w(D_3)$

proof: 1)

D' D

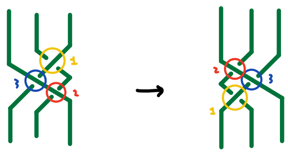
$-$ve crossing

. Then D' has one more crossing than D and is $-$ve.

2)

D' has 2 more crossings with opposite signs

3)



no matter what orientations are, sign of c_i = sign of c_i'.

Thm: if D is a link diagram, then

$$\bar{V}(D) := (-A^3)^{-w(D)} \langle D \rangle$$ is invariant under Reidermeister moves.

proof: 1) $\bar{V}(D_1') = (-A^3)^{-w(D_1)+1}(-A^{-3})\langle D_1' \rangle$

$$= (-A^3)^{-w(D_1)} \langle D_1 \rangle = \bar{V}(D_1)$$

and $\langle D_2 \rangle$ & $\langle D_3 \rangle$ are invariant under RII and RIII. $\square$

Dfn: An oriented, n-component link in $\mathbb{R}^3$ is an isotopy class of embeddings $i : \coprod_{j=1}^{n} S^1 \hookrightarrow \mathbb{R}^3$.

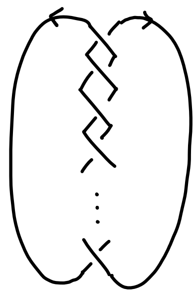
The same proof as knots shows links as diagrams, and diagrams of a link are related by Reidemeister moves.

Dfn: if L is an oriented link, $\bar{V}(L) := (-A^3)^{-w(D)} \langle D \rangle$, where D is any diagram of L, is the unnormalized Jones polynomial of V.

Ex: $\bar{V}(\bigcirc) = B = -A^{-2} - A^2$.

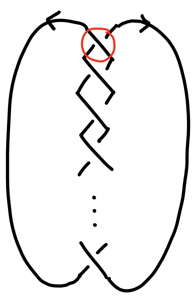
Corollary: If D is a diagram of the unknot, then $\langle D \rangle = (-A^3)^{w(D)} B$.

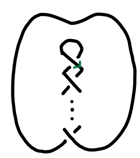
Example: The negative $(2,n)$-torus link is represented by D_n:



n crossings

Resolving:

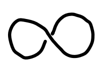


$\langle D_n \rangle = A^{-1} \langle D_{n-1} \rangle + A \langle$  $\rangle$

all +ve crossings now

$$= A^{-1} \langle D_{n-1} \rangle + A \underbrace{(-A^3)^{n-1}}_{} B$$

So we have a recursive relation for this bracket.

So $\langle D_1 \rangle = -A^{-3} B$ $\quad \langle$  $\rangle = -A^{-1-2} B$

$\Rightarrow \langle D_2 \rangle = (-A^{-4} - A^4) B \quad = -A^{-2-2} B (1 + A^8)$

$\Rightarrow \langle D_3 \rangle = (-A^{-5} - A^3 + A^7) B \quad \dots$

$\vdots$

$\langle D_n \rangle = -A^{-n-2} (1 + A^8 - A^{12} + A^{16} + \dots \pm A^{4n}) B$

$w(D_n) = -n$, so $\bar{V}(T(2,-n)) = (-A^3)^{+n} (-A^{-n-2}) (1 + A^8 - A^{12} + \dots \pm A^{4n}) B$

Corollary: $T(2,-n) = T(2,-m) \Rightarrow n = m$.

This is a family of infinitely many different knots.

Better normalization

The normalized Jones polynomial of L is $V_L(q) = \dfrac{\bar{V}(L)}{B} = \left. \dfrac{\bar{V}(L)}{\bar{V}(\bigcirc)} \right|_{q = -A^{-2}}$

Example: $V_L(\bigcirc) = 1$

$V_L(T(2,-n)) = q^{1-n} (1 + q^{-4} - q^{-6} + \dots \pm q^{-2n})$

1.2 Operations on knots / links

Orientation reversal

$r : \coprod_{i=1}^{n} S^1 \to \sqcup S^1$ reverses orientation on each component.

So $r(L) = L \circ r : \sqcup S^n \hookrightarrow \mathbb{R}^3$

Effect on signs:

So $w(r(D)) = w(D)$

and $\langle r(D) \rangle = \langle D \rangle$ since $\langle \; \rangle$ is indep. of orientation

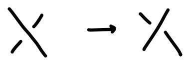
$\Rightarrow$ $\bar{V}(r(L)) = \bar{V}(L)$ and $V(r(L)) = V(L)$

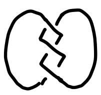
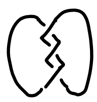
reverse on one but not all: multiply by power of A (or power of q resp) think about.

Mirror:

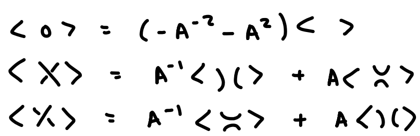
The mirror of L is $\rho \circ L$, where $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is a reflection

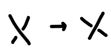
For diagrams, use $\rho(x, y, z) = (x, y, -z)$



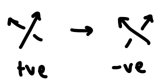
E.g. if $K =$  -ve trefoil $\quad \bar{K} =$  +ve trefoil.

Since



$\Rightarrow$ $\langle \; \rangle$ is invariant under the operation of simultaneously sending  and $A \mapsto A^{-1}$

I.e. $\langle \bar{D} \rangle = \langle D \rangle |_{A \mapsto A^{-1}}$

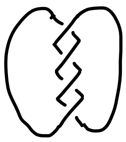
On signs:  $\Rightarrow$ $w(\bar{D}) = -w(D)$

$\Rightarrow$ $\bar{V}(\bar{L}) = \bar{V}(L)|_{A \mapsto A^{-1}}$, equivalently $V(\bar{L}) = V(L)|_{q \mapsto q^{-1}}$

$V(T(2,n)) = q^{n-1}(1 + \dots \pm q^{2n})$

$n = 2$, then odd powers appear.

e.g. $T(2,n) = \overline{T(2,-n)}$



$T(2,4)$

So $V(T(2,n)) = q^{n-1}(1 + q^4 - q^6 \dots \pm q^{2n})$

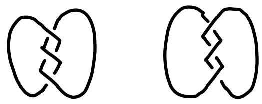
Positive torus knots have positive powers of q in the Jones polynomial by our normalization.

$\Rightarrow$ $T(2,n) \neq T(2,-n)$ $\forall n > 1$.

When $n = 1$, $U = T(2,1) = T(2,-1)$

Disjoint union

Diagrams $D_1, D_2 \to D_1 \sqcup D_2$: 

e.g. $T(2,-3) \sqcup T(2,3)$: 

If $L_1 : \coprod^n S^1 \hookrightarrow \mathbb{R}^3 \subset S_1^3$, $L_2 : \coprod^m S^1 \hookrightarrow \mathbb{R}^3 \subset S_2^3$

Then $L_1 \sqcup L_2 : \coprod^n S^1 \sqcup \coprod^m S^1 \hookrightarrow S_1^3 \# S_2^3 = S^3$

$\Rightarrow$ $L_1 \sqcup L_2 : \coprod^{n+m} S^1 \hookrightarrow S^3$

Note: $\langle D_1 \sqcup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle$

proof: by induction on # of crossings in D_2 skein relation

Writhe : $w(D_1 \sqcup D_2) = w(D_1) + w(D_2)$

$\Rightarrow$ $\bar{V}(L_1 \sqcup L_2) = \bar{V}(L_1)\bar{V}(L_2)$ and $V(L_1 \sqcup L_2) = V(L_1)V(L_2)\underbrace{V(0)}_{= (q^{-1}+q)}$ ← check this.

$$V(L_1 \sqcup L_2) = \left.\frac{\bar{V}(L_1 \sqcup L_2)}{\bar{V}(0)}\right|_{q=-A^{-2}}$$

$$= \left.\frac{\bar{V}(L_1)\bar{V}(L_2)}{\bar{V}(0)}\right|_{q=-A^{-2}}$$

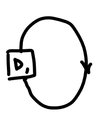
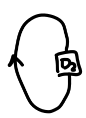
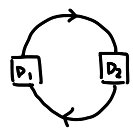
$$= \left.\frac{\bar{V}(L_1)\bar{V}(L_2)\bar{V}(0)}{\bar{V}(0)\bar{V}(0)}\right|_{q=-A^{-2}}$$

$$= V(L_1)V(L_2)\bar{V}(0)\big|_{q=-A^{-2}}$$

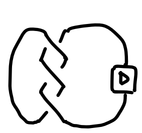
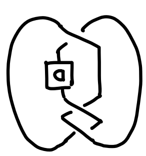
Connected sum.

If $K_1 : S_1^1 \hookrightarrow \mathbb{R}^3 \subseteq S_1^3$, $K_2 : S_2^1 \hookrightarrow \mathbb{R}^3 \subseteq S_2^3$ are knots, I get

$$\begin{array}{ccc} K_1 \# K_2 : S_1^1 \# S_2^1 & \hookrightarrow & S_1^3 \# S_2^3 \\ \cong & & \cong \\ S^1 & \hookrightarrow & S^3 \end{array}$$ is another knot

on diagrams: D_1 =  , D_2 =  , then $D_1 \# D_2$: 

This does not depend on # pt.

e.g.  $\sim$ 

Exercise: $V(K_1 \# K_2) = V(K_1)V(K_2)$

To see knots: KnotInfo or Knot Atlas

1.3. Crossing number

$\langle D \rangle = \sum_{v \in \{\pm 1\}^n} A^{\Sigma v_i} B^{|D_v|} = \sum_v \langle D \rangle_v$ where $\langle D \rangle_v := A^{\Sigma v_i} B^{|D_v|}$ remember $B = -A^{-2} - A^2$

Let $M(D)$ = maximum power of A in $\langle D \rangle$
$m(D)$ = minimum power of A in $\langle D \rangle$
$M_v(D)$ = maximum power of A in $\langle D \rangle_v = \Sigma v_i + 2|D_v|$
$m_v(D)$ = minimum power of A in $\langle D \rangle_v = \Sigma v_i - 2|D_v|$.

$|D_v|$ = # of components in D after resolving via $v \in \{\pm 1\}^n$

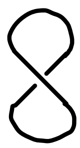
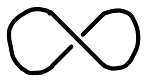
If $v, v' \in \{\pm 1\}^n$, say $v \le v'$ if $v_i \le v_i' \ \forall i$

$v_+ = (+1, \dots, +1)$, $v_- = (-1 \dots -1)$, then $v_- \le v \le v_+ \ \forall v \in \{\pm 1\}^n$.

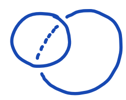
Say $v <_j v'$ if $v_j = -1$, $v_j' = +1$ and $v_i = v_i' \ \forall i \ne j$

Lemma: if $v <_j v'$, then $|D_{v'}| = |D_v| \pm 1$.

proof: Let $D_{\hat{v_j}}$ be the diagram obtained by resolving all crossings according to $v_i = v_i'$ except j^{th} crossing. Then $D_{\hat{v}_j}$ has one crossing, and must look like

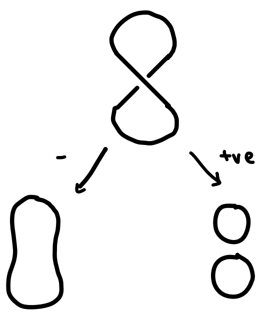
a)  (∞) or b) 

note: crossing will be in one component, not the crossing of two components, otherwise we'd have two crossings, e.g.



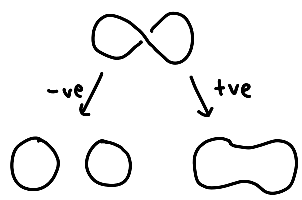
plus a bunch of circles. Resolving according to $v_i = -1$ and $v_i' = 1$,

(a)



$\Rightarrow |D_{v'}| = |D_v| + 1$

Similarly in b), $|D_{v'}| = |D_v| - 1$.



$\square$

Proposition: For all $v \in \{\pm 1\}^n$, $M_v(D) \le M_{v_+}(D)$, and $m_v(D) \ge m_{v_-}(D)$.

proof: If $v <_j v'$, then $M_v(D) = \Sigma v_i + 2|D_v|$, and $M_{v'}(D) = \Sigma v_i' + 2|D_{v'}| \ge \Sigma v_i' + 2(|D_v| - 1)$
$= M_v(D)$

$\left(\Sigma v_i' + 2|D_v| - 2 = \sum_{i \ne j} v_i' + 1 + 2|D_v| - 2 = \sum_{i \ne j} v_i - 1 + 2|D_v| = \sum_i v_i + 2|D_v|\right)$ $M_{v'}(D) \ge M_v(D)$

For any v, we can find a chain $v <_{i_1} v_1 <_{i_2} v_2 \dots <_{i_k} v_+$. $\Rightarrow M_v(D) \le M_{v_1}(D) \le \dots \le M_{v_+}(D)$
and similarly for second statement. turn negatives into positives

$\square$

Cor: $M(D) \le M_{v_+}(D)$ and $m(D) \ge m_{v_-}(D)$

pf: $\langle D \rangle = \sum_v \langle D_v \rangle$

$\square$

So $M(D) - m(D) \le M_{v+}(D) - m_{v-}(D)$ by previous proposition

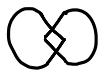
$= (n + 2|D_{v+}|) - (-n - 2|D_{v-}|)$ by definition

$= 2n + 2(|D_{v+}| + |D_{v-}|)$

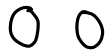
where $n = \#$ of crossings in D.

Say D is connected if the space of the underlying plane curve (forgetting over and under crossings) is connected.

e.g.



connected



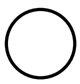
not connected

Lemma: if D is a connected planar diagram with n crossings,

$$|D_{v+}| + |D_{v-}| \le n+2$$

proof: by induction on n:

$n=0$



$|D_{v+}| = |D_{v-}| = 1$

In general, choose a crossing (jth) of D. Let D^- and D^+ be the diagrams obtained by resolving that crossing. At least one of D^- and D^+ is connected since D is. Suppose it's D^-. Then $|(D^-)_-| + |(D^-)_+| \le (n-1)+2$ by induction. Also $(D^-)_+ <_j D_+$ $\Rightarrow$ $|D_+| \le |(D^-)_+| + 1$ by our lemma. $|D_+| = |(D^-)_+| \pm 1$

$\Rightarrow$ $|(D^-)_-| + |D_+| \le (n-1)+2+1 = n+2$. But $(D^-)_- = D_-$, so

$|(D^-)_-| + |D_+| \le |(D^-)_-| + |(D^-)_+| + 1 \le (n-1)+2-1$

$\Rightarrow |D_-| + |D_+| \le n+2.$

If D is a planar diagram, let $c(D)$ be the number of crossings in D.

Cor: if D is a connected diagram with $\overset{c(D)}{\overset{\shortparallel}{n}}$ crossings, then $M(D) - m(D) \le M(\langle D\rangle_{v+}) - m(\langle D\rangle_{v-})$, where $\langle D\rangle_v = A^{\Sigma v_i}(-A^{-2} - A^2)^{|D_v|}$. Then

$M_{v+}(D) - m_{v-}(D)$

$\Sigma(v+)_i = c(D)$ $\qquad$ $\Sigma(v-)_i = -c(D)$

$$\begin{aligned} M(\langle D\rangle_{v+}) - m(\langle D\rangle_{v-}) &= (c(D) + 2|D_{v+}|) - (-c(D) - 2|D_{v-}|) \\ &= 2c(D) + 2|D_{v+}| + 2|D_{v-}| \\ &\le 2c(D) + 2(c(D)+2) \\ &= 4c(D) + 4 \end{aligned}$$

Definition: say L is **nonsplit** if every diagram D representing L is connected. That is, $L \ne L_1 \sqcup L_2$ for L_1, L_2 nonempty.

Definition: If L is a link, its **crossing number** is $c(L) = \min\{c(D) \mid D \text{ is a diagram of } L\}$

Write $M_q(v(L))$ = maximal power of q in $v(L)$, and

$m_q(v(L))$ = minimal power of q in $v(L)$

If L is split, then $L = L_1 \sqcup L_2$ so

$v(L) = v(L_1)v(L_2)v(O)$

$\Rightarrow v(O) \mid v(L)$.

Thm: (Kauffman) if L is a nonsplit link, then

$$c(L) \geq \frac{1}{2}\left(M_q(v(L)) - m_q(v(L))\right)$$

pf: if D is a diagram of L, D is connected and $\quad v(L) = \left.\dfrac{\bar{v}(L)}{-A^{-2}-A^{2}}\right|_{q=-A^{-2}}$

So $\quad M_q(v(L)) - m_q(v(L)) = \frac{1}{2}\left(M_A(\bar{v}(L)) - m_A(\bar{v}(L)) - 4\right)$

$$\leq \frac{1}{2}\left(4\,c(D) + 4 - 4\right)$$

$$= 2\,c(D)$$

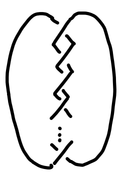
Example: $L = T(2,n)$

$$v(L) = q^{n-1}\left(1 + q^4 - q^6 + q^8 \dots \pm q^{2n}\right)$$

so $\quad M_q(v(L)) - m_q(v(L)) = 2n$

$\Rightarrow \quad c(D) \geq n$

But $\quad T(2,n)$



has exactly n crossings

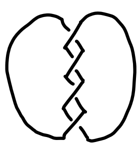
$\Rightarrow \quad c(T(2,n)) = n$

N.B. $T(2,n)$ is nonsplit, since $\quad (q + q^{-1}) \nmid v(L)$

1.4. Alternating Knot

Dfn: a diagram D is alternating if as we traverse D, crossings alternate between over and under

e.g. $\quad T(2,4)$:



is alternating, as is every $T(2,n)$

e.g.

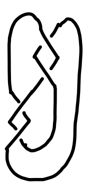
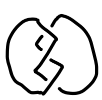


figure 8 knot
is also alternating

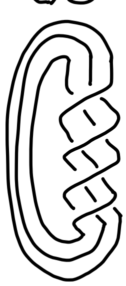
L is alternating if it has an alternating diagram, and is nonalternating otherwise

non example:



not alternating, but boring (unknot)

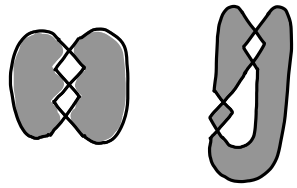
$T(3,4)$



not alternating.

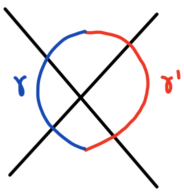
Dfn: if D is a planar diagram, a checkerboard coloring of D is an assignment of colors (black or white) to each region of the complement of Dgraph such that the colors on either side of every edge differ.

Dgraph is the underlying planar 4-valent graph of D

Example: 

lemma: every planar diagram has exactly 2 checkerboard colorings (related by switching color)

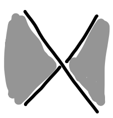
proof: Fix a region R_0. For any other region R, pick a path γ from R_0 to R, which misses vertices of Dgraph. Then the mod 2 intersection # of γ with Dgraph determines the color of R from the color of R_0. This does not depend on the choice of γ since every vertex of Dgraph has even valency (4)



mod 2 ∩# of loop is zero, so they must be same.
(4 mod 2)

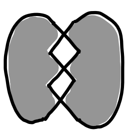
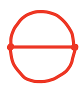
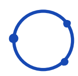
□

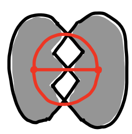
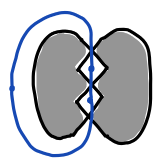
Given a checkerboard coloring of D, we can form two new planar graphs B(D), W(D), where vertices of B(D) are the black regions, and edges are the crossings of D.

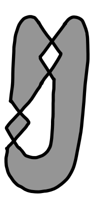
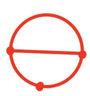
e.g.  then B(D) =

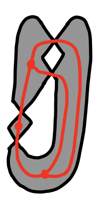
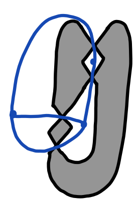
edge goes through crossing

Similarly for W(D) with white regions.

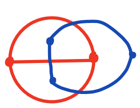
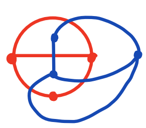
e.g 1)  then B(D) :  and W(D) = 

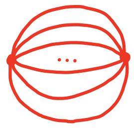
e.g. 2)  then B(D) :  and W(D) = 

Observe that B(D) and W(D) are dual planar graphs. That is, vertices of W(D) ⇔ complementary regions of B(D) and edges of W(D) ⇔ edges of B(D)

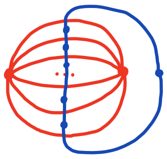
e.g. 1)  and 2) 

Example: $T(2,n)$: $B(D)$ =

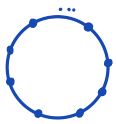


2 vertices
n edges

To find $W(D)$:

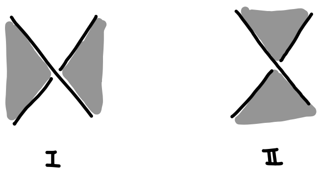


=



n vertices
n edges

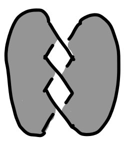
At a crossing:



2 possibilities

We'll say that a coloring is **consistent** if all crossings are I or all crossings are II.

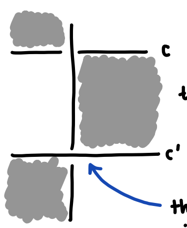
Taking:



this is consistent.

Lemma: if D is a connected planar diagram, then D is **consistent iff D is alternating.**

proof:



type 1

this crossing has type I iff crossings alternate as I go from c to c'

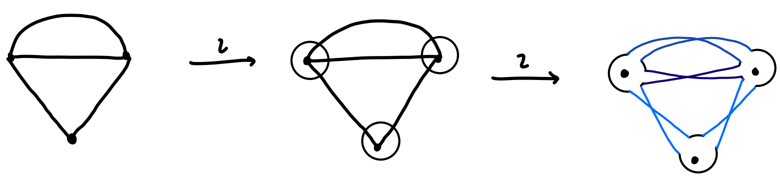
D is connected ⇒ I get to all crossings using this method. So D alternating ⇔ D is consistent. □

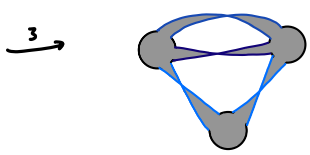
Conversely, given a planar graph B, there is a unique alternating diagram D with $B(D) = B$.

To construct D,

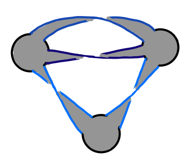
1) Start with a disk around each vertex of B
2) Add a crossing along each edge (this determines D graph)
3) Choose all crossings to be consistent with coloring.

e.g.





+ choose crossings to be consistent =



=

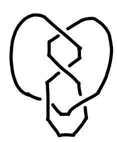
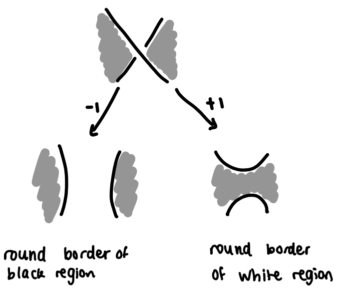


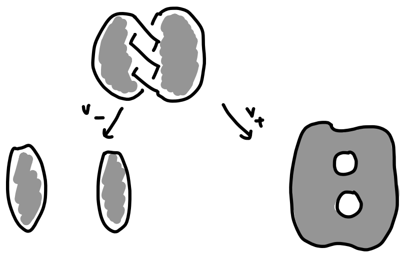
figure 8 knot.

If every crossing is type I, then



So components of D_{v_-} are boundaries of black regions, and components of D_{v_+} are boundaries of white regions

E.g.



Lemma: if D is a connected alternating diagram, then $|D_{v_-}| + |D_{v_+}| = c(D) + 2$

proof: $|D_{v_-}|$ = # of vertices in $B(D)$, $|D_{v_+}|$ = # of vertices in $W(D)$
= # faces for $B(D)$

So $B(D)$ lies on a sphere, so $V - E + F = 2$.

$$\Rightarrow |D_{v_-}| - E + |D_{v_+}| = 2$$

and E = number of edges = # of crossings = $c(D)$

Hence,

$$|D_{v_-}| + |D_{v_+}| = c(D) + 2.$$

$\square$

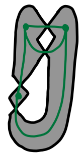
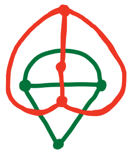
01/02/2022

Last lecture:

prop: There's a bijection

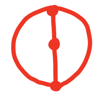
$$\left\{ \begin{array}{c} \text{Connected alternating} \\ \text{planar diagrams} \end{array} \right\} \qquad \left\{ \begin{array}{c} \text{Connected planar} \\ \text{graphs} \end{array} \right\}$$

$D \longmapsto B(D)$, where we use type 1 coloring

Remark: mirror (D) is equivalent to taking planar dual (switches black + white graphs) e.g. e.g. 2)  

black graph
white graph

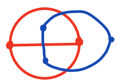
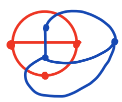
Rem: black graph of $\bar{D}$

 $\sim$ 

over S^2

$\Rightarrow D = \bar{D}$
(i.e. K is amphichiral, $K = \bar{K}$)

Observe that $B(D)$ and $W(D)$ are dual planar graphs. That is, vertices of $W(D) \Leftrightarrow$ complementary regions of $B(D)$ and edges of $W(D) \Leftrightarrow$ edges of $B(D)$

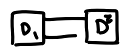
e.g. 1)  and 2) 

The set D_{v-} = boundary of the black regions,
D_{v+} = boundary of the white regions

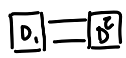
$\Rightarrow$ By Euler, $|D_{v+}| + |D_{v-}| = c(D) + 2$

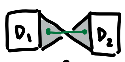
Example:  alternating diagram of unknot, but does not have minimal crossing #.

Dfn: A crossing c of D is **nugatory** if D looks like

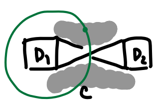
1)  $\sim$ 

or

2)  $\sim$ 

1) $B(D)$ has a bridge  , removing it disconnects the graph.
$W(D)$ has a loop

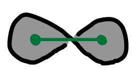
still assuming all crossings are type I.

2) $B(D)$ has a loop  (edge with both endpoints the same vertex.
$W(D)$ has a bridge

We say D is reduced if it has no nugatory crossing
$\Leftrightarrow$ $B(D)$ has no loops and no bridges
$\Leftrightarrow$ $B(D)$ and $W(D)$ have no loops.

Lemma: if D is a reduced, alternating diagram, then $m(D) = m(\langle D\rangle_{v-}) = -n - 2|D_{v-}|$
and $M(D) = M(\langle D\rangle_{v+}) = n + 2|D_{v+}|$.

proof: Suppose that $v_- <_i v'$. The diagram obtained by resolving all but the i^{th} crossing looks like

(and a bunch of circles) A) 

you can get to a bridge by resolving, but you can't get to a loop by resolving.

Since D is reduced, no edge of $B(D)$ is a loop, so diagram looks like A) instead of 

↳ green edge remaining is from original graph.

So A) $\Rightarrow |D_{v_-}| = |D_{v'}| + 1$ from our lemma
$\Rightarrow m(\langle D\rangle_{v-}) < m(\langle D\rangle_{v'})$
$\Rightarrow m(\langle D\rangle_{v-}) < m(\langle D\rangle_v)$ for all v

Since $\langle D\rangle = \Sigma \langle D\rangle_v$, $m(D) = m(\langle D\rangle_{v-})$. Similarly for $M(D)$ and D_{v+} (use white graph + no loops) $\square$

Corollary: if D is a reduced alternating diagram, then

$$M(D) - m(D) = M(\langle D\rangle_{v+}) - m(\langle D\rangle_{v-}) = n + 2|D_{v+}| - (-n - 2|D_{v-}|)$$
$$= 2(n + |D_{v+}| + |D_{v-}|)$$
$$\leq 2n + 2(2n+2) = 4n+4$$

$\square$

where $n = \#$ crossings in D.

(First Tait Conjecture)
Theorem: if L is a nonsplit link, and D is a reduced alternating diagram of L, then $c(L) = c(D)$.

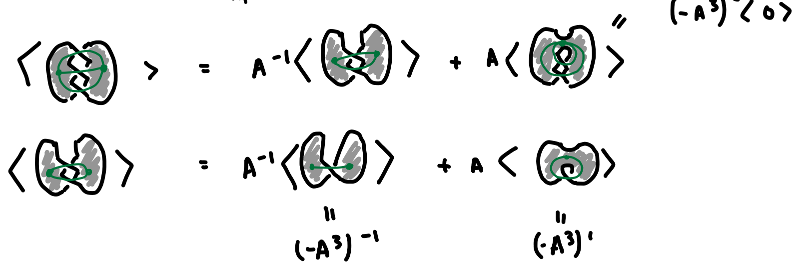
proof: if D' is any diagram, then

$$c(D') \geq \tfrac{1}{2}\left(M_q(V(L)) - m_q(V(L))\right)$$
$$\geq \tfrac{1}{2}\left(\tfrac{1}{2}(M(D) - m(D) - 4)\right) = 2c(D) \text{ by Corollary}$$

$\square$

1.5. Maximal Trees

Recall how we compute

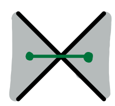


want to think about what happens to the black graph.

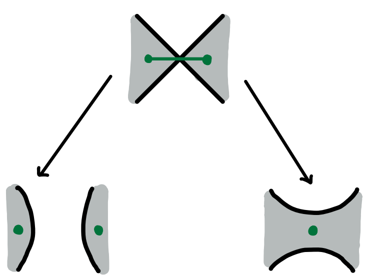
Operations on Planar Graphs

If e is an edge of G, (connected planar graph) which is neither a loop nor a bridge, then $G\setminus e$ (remove e) and G/e (squash) (collapse e to a point) are also connected planar graphs

If D is a conneted planar diagram (not necessarily alternating) w/ a checkerboard coloring and c is a crossing of D

diagram 

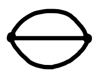
Assume c is non-nugatory $\Leftrightarrow$ corresponding edge e is neither a loop nor a bridge. Then we have two resolutions



$B(D_{)(}) = B(D)\setminus e$

$B(D_{\asymp}) = B(D)/e$

Definition: if G is a graph, a **maximal tree** of G is a subgraph which is a tree and contains every vertex of G.

Ex:  has 3 maximal trees: , , and 

Definition: A connected planar graph G is **small** if every edge is either a loop or a bridge.

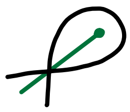
Proposition: If G is small, then

a) G has a **unique maximal tree**

b) If D is any planar diagram, with $B(D) = G$, then D can be **unknotted** using only RI moves ($\Rightarrow$ is the unknot)

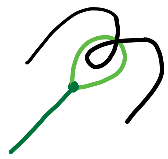
proof: (a) no loop can be an edge in a maximal tree. Let G' be the result of deleting all loops from G. Every edge of G' is a bridge $\Rightarrow$ only maximal tree is G' itself.

(b) G' is a tree. Choose v which is a leaf of G'. If v has no loops attached to it in G, then D is as shown.



So D can be simplified by an RI move, reducing # of edges in graph. Then do induction.

If v has loops, e.g. , find an innermost loop attached to v, then it looks like



So D can be simplified by an RI move, reducing # of edges in graph.

After simplification, the new graph still has only bridges and loops. By induction, the corresponding diagram will be reduced to a single vertex, which has associated knot the unknot.

□

Recall a subgraph G' is a maximal tree iff.

a) G' is connected

b) $V_{G'} = V_G$ (*)

c) $\chi(G') = 1$ (euler characteristic)

$\chi(G) = V - E + F$

$= n - (n-1) + 0 = 1$

Say an edge e is **interesting** if it is **neither** a loop nor a bridge. If e is interesting, then we have two operations $G \setminus e$ (remove e), and G/e (collapse e) are connected planar graphs.

deleting does not disconnect graph

Definition: let $\mathcal{U}(G) = \{ T \subset G : T \text{ is a maximal tree} \}$

neither loop nor bridge

Lemma: if e is an interesting edge of G, then there's a bijection

$$\mathcal{U}(G) \longleftrightarrow \mathcal{U}(G\backslash e) \sqcup \mathcal{U}(G/e)$$

$$T \longmapsto \begin{cases} T \subset G\backslash e & \text{if } e \notin T \\ T/e \subset G/e & \text{if } e \in T \end{cases}$$

proof: using Criteria (*), it's easy to see that the image of T is a maximal tree, and the inverse is given by

$$T \in \mathcal{U}(G\backslash e) \longrightarrow T \subset G$$

$$T \in \mathcal{U}(G/e) \longrightarrow T \cup e \quad \text{adding back the edge}$$

$\square$

Standing assumptions:

D is a connected diagram of a link L, $G = B(D)$, and write $\langle D\rangle' := \dfrac{\langle D\rangle}{\langle O\rangle} = \dfrac{\langle D\rangle}{-A^2-A^{-2}} = (-A^3)^{w(D)} V(L)\Big|_{q=-A^{-2}}$

Proposition: $\langle D\rangle' = \sum\limits_{T\in\mathcal{U}(G)} A^{f(T)}\langle D_T\rangle'$ $\qquad f: T \to \mathbb{Z}$ some function (not too important)

where $G_T = B(D_T)$ is small (has no interesting edges)

proof: By induction on number of crossings of D.

$D = O$ is obvious. Given a general D, if $G = B(D)$ is small then I'm done. Otherwise choose an interesting edge of D and resolve the corresponding crossing. Then

$$\langle D\rangle' = A^{-1}\langle D_{)(}\rangle' + A\langle D_{\asymp}\rangle'$$

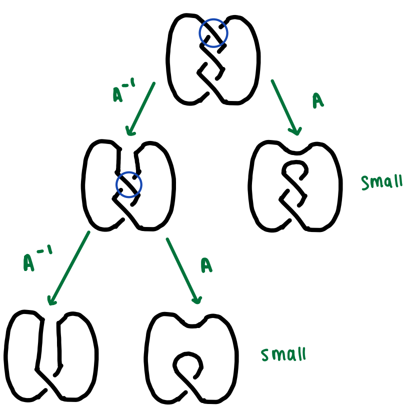
$$\{B(D_{)(}),\ B(D_{\asymp})\} = \{G/e,\ G\backslash e\}$$

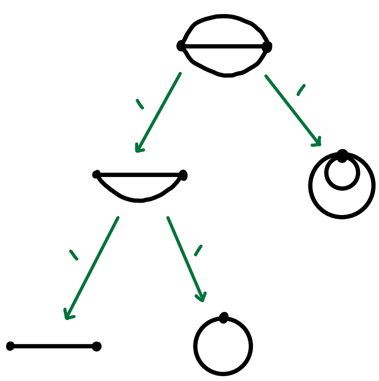
By induction applied to $D_{)(}$ and $D_{\asymp}$, $\langle D\rangle' = A^{\pm 1}\sum\limits_{T\in\mathcal{U}(G\backslash e)} A^{f_1(T)}\langle D_T\rangle' + A^{\mp 1}\sum\limits_{T\in\mathcal{U}(G/e)} A^{f_2(T)}\langle D_T\rangle'$

$\square$

By lemma, $\Rightarrow$ $\langle D\rangle' = \sum\limits_{T\in\mathcal{U}(G)} A^{f(T)}\langle D_T\rangle'$.

Best way to think about this is via binary trees





Corollary: if $V(L) = q^k\left(\sum a_i q^{2i}\right)$, then $\sum |a_i| \leq \# \mathcal{U}(G)$

proof: D_T small $\Rightarrow$ $\langle D_T \rangle' = (-A^3)^{w(D_T)}$ $B(D_T)$ small $=$ D_T represents the unknot.

$\Rightarrow \langle D \rangle' =$ sum of $\# \mathcal{U}(G)$ terms that look like $\pm A^{g(T)}$

$$\langle D \rangle' = \sum_{T \in \mathcal{U}(G)} A^{f(T)} \langle D_T \rangle'$$

Then you multiply by factors of A to get $V(L)$ which does nothing to the coefficients

Definition: polynomial $p(q) = q^k\left(\sum a_i q^{2i}\right)$ is **alternating** if $a_i a_{i+1} \leq 0 \ \forall i$.

Example: $V(T(2,n)) = q^{n-1}\left(1 + q^4 - q^6 + q^8 - q^{10} + \dots \pm q^{2n}\right)$ is an alternating polynomial

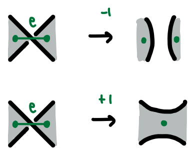
Theorem: if D is a connected alternating diagram, then

a) $V(L)$ is alternating and

b) $\sum |a_i| = \left| V(L)|_{q^2=-1} \right| = \# \mathcal{U}(G)$

recall: alternating diagram means crossings alternate between over and under as you traverse the knot

proof: Suppose D has a type I coloring, so every crossing looks like . Then the resolutions $D_{)(}$ and $D_{\asymp}$ are both alternating of type I coloring:



$D_{)(}$: $\xrightarrow{-1}$ $G\backslash e$

$D_{\asymp}$: $\xrightarrow{+1}$ G/e

if $V(L)$ is alternating,

Consider $h(G) = V_G + \chi(G)$. Under our operations

A^{-1} $G \to G\backslash e$, then
$V \mapsto V$
$\chi \mapsto \chi + 1$
$\Rightarrow h \mapsto h+1$

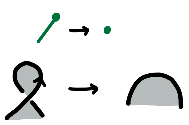
A^{+1} $G \to G/e$, then
$V \mapsto V-1$
$\chi \mapsto \chi$
$\Rightarrow h \mapsto h-1$

The net result:

$$\langle D \rangle' = \sum A^{h(G) - h(G_T)} \langle D_T \rangle'$$

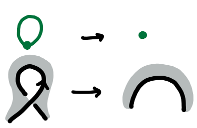
Simplify D_T using RI moves. To simplify D_T', either

- remove a leaf



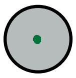
so that $w \to w+1$
and $V \to V-1$, $\chi \to \chi$
$\Rightarrow h \to h-1$

- or remove a loop.



so that $w \to w-1$
and $V \to V$, $\chi \to \chi+1$
$\Rightarrow h \to h+1$

So change in writhe w is opposite to change in h, so



$\left.\begin{matrix} w=0 \\ \chi=1 \quad \Rightarrow h=2 \\ v=1 \end{matrix}\right\}$ $w = 2-h$ by induction

In summary, $\langle D\rangle' = \sum_{T\in\mathcal{M}(G)} A^{h(G)-h(G_T)} \langle D_T\rangle'$

$$= \sum_T A^{h(G)-h(G_T)} (-A^3)^{w(D_T)}$$

$$= \sum_T A^{h(G)-h(G_T)} (-A^3)^{2-h(G_T)}$$

$$= A^{h(G)+6} \sum_T (-1)^{h(G_T)} (A^4)^{-h(G_T)}$$

Which is precisely an alternating polynomial in A^4 $\Rightarrow$ $V(L)$ is also alternating.

a) $\Rightarrow$ b) is easy (look at coefficients of polynomial) $\square$

Definition: if L is a link, it's determinant is $\det(L) := \left|V(L)|_{q^2=-1}\right|$.

Theorem $\Rightarrow$ if L is alternating, $\det(L) = \#\mathcal{M}(G)$ where $G = B(D)$ is any nonsplit alternating diagram of L.

Note that if L is split, $(q+q^{-1}) \mid V(L)$ $\Rightarrow$ $\det(L) = 0$. (split $\Rightarrow$ can resolve into a link with disjoint unknot component)

Corollary: if D is a nonsplit, connected, alternating diagram of L, then L is nonsplit.

proof: $\det L = \#\mathcal{M}(B(D)) > 0$ $\square$

Open question: is the $c(K_1 \# K_2) = c(K_1) + c(K_2)$?

True if K_1, K_2 are alternating

Best general bound (Lackenby)

$$c(K_1 \# K_2) \geq \frac{c(K_1)+c(K_2)}{152}$$

2] Alexander Polynomial

2.1 Knot Exterior

Tubular neighbourhood Theorem: if $N \subset M$ is an embedded submanifold with normal bundle $\nu_{N/M}$, then there is an embedding $j: D(\nu_{N/M}) \hookrightarrow M$ with $j \circ s_0 = $ inc. $N \hookrightarrow M$

Idea of proof: use exponential map $\exp: T_x M \longrightarrow M$ which sends $v \in T_x M$ to $\gamma_v(1)$, where γ_v = unique geodesic with $\gamma(0) = x$ and $\gamma'(0) = v$ (we pick any Riemannian metric). Consider $\exp|_{\nu_{N/M}}$, $\nu_{N/M} = TN^{\perp} \subset TM$, and compute $d\exp = $ id. Define $j(v) = \exp(v)$ and use inverse function thm to see j is an embedding.

An aside: Links in S^3

Definition: an oriented, n-component link in S^3 is an isotopy class of embeddings $L: \coprod^n S^1 \hookrightarrow S^3$.

Note $\mathbb{R}^3 \subset S^3$ (one point compactification) so we get a map by inclusion

$$\Psi : \{\text{oriented links in } \mathbb{R}^3\} \longrightarrow \{\text{oriented links in } S^3\}$$

Standard transversality arguments show that

1) Ψ is surjective: any $L \hookrightarrow S^3$ is isotopic to L' which misses ∞

∞ is a 0 dim. submanifold of S^3 and L a 1 - dim submanifold

2) Ψ is injective: any $L \times I \hookrightarrow S^3$ generically misses ∞

$L \times I$ is 2-dim and ∞ 0-dim

Links in $\mathbb{R}^3 \leftrightarrow$ links in S^3

Return to TNT:

Suppose $N \subset M$ is a smooth submanifold, and let $\nu = \nu_{M/N}$ be the normal bundle, with $s_0 : N \to \nu$ the zero section.

Definition: $j: D(\nu) \hookrightarrow M$ is a tubular closed neighbourhood of N if

a) $j \circ s_0 = \mathrm{id}_N$

b)
$$\begin{array}{ccccc} dj|_{s_0(x)} & : & T_{s_0(x)}\nu & \to & T_x M \\ \| & & \cong & & \cong \\ \mathrm{id} & : & T_x N \oplus \nu_x & \to & T_x N \oplus \nu_x \end{array}$$

Tubular Neighbourhood Theorem: if $N \subset M$ is a smooth submanifold, then

a) $\exists$ a tubular neighbourhood $j: D(\nu) \hookrightarrow M$

b) if $j, j' : D(\nu) \hookrightarrow M$ are two tubular neighbourhoods, then $j \sim_i j'$ (isotopic)

idea of proof: a) define $j(\overset{\text{vector}}{v}) = \exp_{\pi(v)}(\varepsilon v)$ where $\exp_x(v) = \gamma(1)$ and γ is the unique geodesic with $\gamma(0) = x$ and $\gamma'(0) = v$ (pick a Riemannian metric) (taking ε sufficiently small)

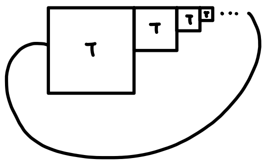
b) Define $\mu_\varepsilon : \nu \to \nu$; $v \mapsto \varepsilon v$. Then $j \sim_i j \circ \mu_\varepsilon$ and $j' \sim_i j \circ \mu_\varepsilon$, so it's enough to prove that $j|_{D_\varepsilon(v)} \sim_i j'|_{D_\varepsilon(v)}$. Argue as in proof $f|_{B_\varepsilon(x)} \sim df|_x$ (similar to first lecture)

Definition: if $N \subset M$ (assume M compact) is a smooth submanifold, and $j: D(\nu) \hookrightarrow M$ is a tubular neighbourhood, the exterior of N is $M - j(D^\circ(\nu))$, denoted E_N.

If $\iota: N \hookrightarrow M$ is an embedding, write $E_\iota = E_{im(\iota)}$. Note E_ι is a compact manifold with boundary, $\partial E_\iota = S(\nu)$. The complement of N is $M - N$, which is a noncompact manifold. We have $M - N \cong E_N \cup_{\partial E_N} \partial E_N \times [0, \infty)$ (using $D^n - \{0\} \cong S^{n-1} \times [0,\infty)$). In particular, $E_N \simeq M - N$

idea: like gluing back everything up until N

Example. Let $w: S^1 \hookrightarrow S^3$ be the "wild embedding" from second lecture



Then $\pi_1(S^3 - im(w))$ is not a finitely generated group $\Rightarrow$ $S^3 - im(w)$ is not homotopy equivalent to a compact 3-manifold with boundary. So w has no tubular neighbourhood. Continuous maps are not your friends! Need smoothness. <u>But</u>: $S^3 - im(w)$ is a smooth 3-manifold. (wild end).

Lemma: if $k: S^1 \hookrightarrow S^3$, then $\nu_{S^3/k}$ is trivial.

proof: $\nu_{S^3/k}$ is a two dimensional vector bundle over S^1. By the clutching construction, such v.b. are in bijection with $\pi_{1-1}(O(2)) = \pi_0(O(2))$ which has two elements. Explicitly,

$$T = I \times \mathbb{R}^2 / (0,v) \sim (1,v)$$
$$M = I \times \mathbb{R}^2 / (0,v) \sim (1, r(v))$$

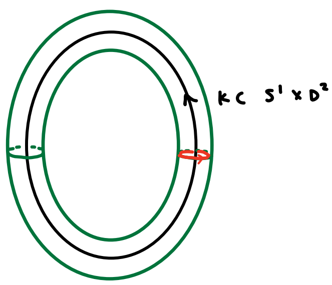
$r: \mathbb{R}^2 \to \mathbb{R}^2$ reflection

Notice that M is nonorientable as a vector bundle $\Rightarrow$ $D(M)$ is nonorientable as a 3-manifold. But S^3 is orientable, so $D(M) \not\hookrightarrow S^3$. So it must be that $\nu_{S^3/k}$ is the trivial bundle.

$\square$

So $D(\nu_{S^3/k}) \cong S^1 \times D^2 =: \nu(k)$ (notation)

Picture:



generator of ker = red curve

$\partial \nu(k) \cong T^2$. The map $i_*: H_1(\partial\nu(k)) \to H_1(\nu(k))$ has $\ker i_* \cong \mathbb{Z} \cong \langle [\partial D^2] \rangle$

$$\begin{array}{ccc} H_1(T_2) & \longrightarrow & H_1(S^1 \times D^2) \\ \wr\wr & & \\ \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \end{array}$$

An orientation on k determines a preferred generator m for $\ker(i_*)$ according to right hand rule (or alt. intersection #: $k \cdot [D^2] = 1$ in $H_*(\nu(k)) \Leftrightarrow k \cdot [\partial D^2] = 1$ in $H_*(\partial\nu(k))$)

This m is called the meridian of k

Proposition: Suppose $L : \bigsqcup^n S^1 \hookrightarrow S^3$ is a link. Then

$$H_*(E_L) = \begin{cases} \mathbb{Z}^{n-1} & \text{if } * = 2 \\ \mathbb{Z}^n & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 0 \end{cases}$$

and $H_1(E_L) = \langle m_1, \dots, m_n \rangle$ where m_i is the meridian of the i^{th} component

proof: $S^3 = E_L \cup_{\partial \nu(L)} \partial \nu(L)$, $\nu(L)$: tubular nhood of L. By lemma, $\nu(L) \simeq \bigsqcup^n S^1 \times D^2$, and $\partial \nu(L) \simeq \bigsqcup^n T^2$ (homeos). Consider the Mayer Vietoris sequence:

$$\underset{\overset{\|}{0}}{H_3(E_L) \oplus H_3(\nu(L))} \to \underset{\overset{\|}{\langle [S^3] \rangle}}{H_3(S^3)} \xrightarrow{\partial} \underset{\overset{\|}{\mathbb{Z}^n = \langle [T_i^2] \rangle}}{H_2(\partial \nu(L))}$$

$$\to H_2(E_L) \oplus \underset{\overset{\|}{0}}{H_2(\nu(L))} \to \underset{0}{\cancel{H_2(S^3)}} \xrightarrow{\partial} H_1(\partial \nu(L)) = \mathbb{Z}^{2n}$$

$$\to H_1(E_L) \oplus \underset{\underset{\mathbb{Z}^n}{\|}}{H_1(\nu(L))} \to \underset{0}{\cancel{H_1(S^3)}} \to \dots$$

Homology of solid torus:

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i \geq 2 \end{cases}$$

Consider that $\partial [S^3] = \oplus [T_i^2] \Rightarrow H_2(E_L) = \mathbb{Z}^n / \langle (1, \dots, 1) \rangle \simeq \mathbb{Z}^{n-1}$

Also $0 \to \mathbb{Z}^{2n} \xrightarrow{i_{1*} \oplus i_{2*}} H_1(E_L) \oplus \mathbb{Z}^n \Rightarrow H_1(E_L) \simeq \mathbb{Z}^n$ and $i_{2*}(m_i) = 0 \Rightarrow H_1(E_L) = \langle m_i \rangle_{i = 1, \dots, n}$. $\square$

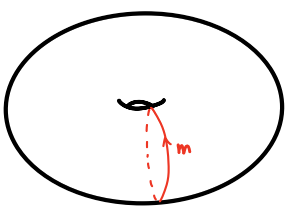
This is disappointing, since the answer doesn't depend on anything except # of components of L.

Let's also compute $H_*(E_K, \partial E_K)$. By LES, ...

2.2. Seifert Surfaces

Suppose $N \subset M$ is an n-dimensional closed, connected, oriented, embedded submanifold with inclusion $i : N \hookrightarrow M$. Then $H_n(N) \cong \mathbb{Z} = \langle [N] \rangle$, and write $[N] = i_*([N]) \in H_n(M)$.

More generally,



$[m] \in H_1(T^2)$ (loop)

Now suppose $(N, \partial N) \hookrightarrow (M, \partial M)$, where N is a compact, connected, oriented n-manifold with boundary. $\partial N \neq \emptyset$. Then LES of $(N, \partial N)$ looks like

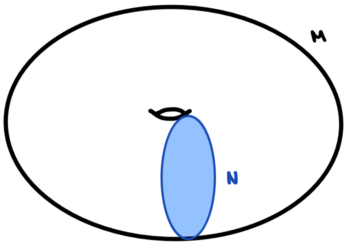
$$\underset{\underset{0}{\|}}{H_n(N)} \to \underset{\underset{\mathbb{Z}}{\|}}{H_n(N, \partial N)} \hookrightarrow H_{n-1}(\partial N) \to H_{n-1}(N)$$

connected with $\partial N \neq \emptyset$

injective by previous group being 0.

$[\partial N]$

Then $H_n(N, \partial N) \cong \mathbb{Z} = \langle [N, \partial N] \rangle$, write $[N, \partial N] = i_*([N, \partial N]) \in H_n(M, \partial M)$.



Solid torus

$[N, \partial N] \in H^2(S^1 \times D^1, S^1 \times S^1)$

We have a commuting map of LES of pairs: $[N, \partial N] \xmapsto{\partial} [\partial N]$ in $H^2(N, \partial N) \xrightarrow{\partial} H^2(\partial N)$

$$\begin{array}{ccccccccc} \cdots \to & H_n(N) & \to & H_n(N, \partial N) & \xrightarrow{\partial} & H_{n-1}(\partial N) & \to \cdots \\ & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & \\ \cdots \to & H_n(M) & \to & H_n(M, \partial M) & \xrightarrow{\partial} & H_{n-1}(\partial M) & \to \cdots \end{array}$$

Hence $\underset{H_n(M, \partial M)}{\partial(\underset{\cap}{[N, \partial N]})} = [\partial N] \in H_{n-1}(\partial M)$

Homology of the Torus:

Proposition: 1) $H_1(T^2) \cong \mathbb{Z}^2$

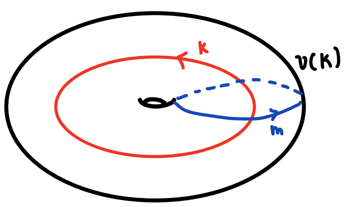
2) if $\alpha \in H_1(T^2)$, $\alpha \in [\gamma]$, where $\gamma : S^1 \hookrightarrow T^2$ is a simple closed curve iff α is primitive (i.e. $\alpha \neq k\beta$ for some $k > 1$), and

3) if $\gamma, \gamma' : S^1 \hookrightarrow T^2$ with $[\gamma] = [\gamma']$, then $\gamma \sim_i \gamma'$.

proof: think about 2. cut torus along γ

3) mapping class groups.

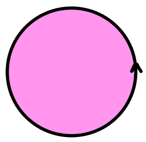
Suppose $K \hookrightarrow S^3$ is an oriented knot. Then $S^3 = E_K \cup_{\partial \nu(K)} \nu(K)$, where $\nu(K) \cong S^1 \times D^2$, so $\partial \nu(K) \simeq S^1 \times S^1 = T^2$



$m = m(K)$ is the meridian of K

Before we've shown that $H_*(E_K) = \begin{cases} \mathbb{Z} & * = 0,1 \\ 0 & \text{otherwise} \end{cases}$, $H_1(E_K) = \langle m \rangle$.

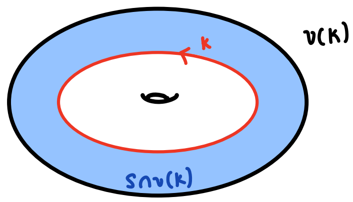
Definition: a **Seifert surface** of K is an embedded, oriented surface $S \hookrightarrow S^3$, with $\partial S = K$ (as oriented manifolds).

Example: unknot : 

Tubular neighbourhood theorem $\Rightarrow$ we can choose $\nu(K)$ such that $(\nu(K), S \cap \nu(K)) \cong (D(\nu_{S^3/K}), TS \cap D(\nu_{S^3/K}))$

S is our Seifert surface

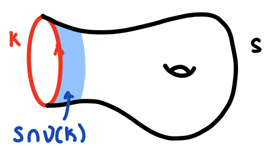
Picture:

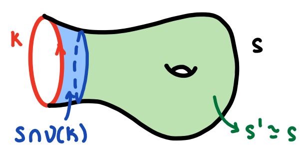


$S \cap \nu(K)$ is an annulus bounded by K and the boundary of $\nu(K)$.

$S \cap \nu(K)$ is a tubular nhood of ∂S in S.

Then $S' := S \cap E_K \cong S$ (homeo)





Prop:
1) $[\partial S'] \in H_1(\partial E_K)$ generates $\ker i_* : H_1(\partial E_K) \to H_1(E_K)$

remember $\dim(S') = 2$ so $\dim(\partial S') = 1$

2) $\langle m, [\partial S'] \rangle$ is a basis for $H_1(\partial E_K)$
3) $H_*(E_K, \partial E_K) \cong \begin{cases} \mathbb{Z} & * = 2,3 \\ 0 & \text{otherwise.} \end{cases}$

and $H_2(E_K, \partial E_K) = \langle [S', \partial S'] \rangle$

proof: LES of $(E_K, \partial E_K)$:

$$\begin{array}{ccccc} \overset{0}{\shortparallel} & & & & \overset{\mathbb{Z}}{\shortparallel} \\ H_3(E_K) & \longrightarrow & H_3(E_K, \partial E_K) & \longrightarrow & H_2(\partial E_K) \\ \overset{0}{\shortparallel} & & & & \\ H_2(E_K) & \longrightarrow & H_2(E_K, \partial E_K) & \longrightarrow & H_1(\partial E_K) \\ & & i_* & & \\ \overset{\mathbb{Z}}{\shortparallel} & & & & \\ H_1(E_K) & \longrightarrow & H_1(E_K, \partial E_K) & \longrightarrow & H_0(\partial E_K) \longrightarrow \cdots \end{array}$$

since $\dim(\partial E_K) = 2$, compact, connected, ... or just $\partial E_K \cong T^2$.

remember: $H_i(E_K) = \begin{cases} \mathbb{Z}^{n-1} & i=2 \\ \mathbb{Z}^{q} & i=1 \\ \mathbb{Z} & i=0 \\ 0 & \text{otherwise} \end{cases}$

$m \in H_1(\partial E_K) \Rightarrow i_*$ surjective.

Top row of LES: $\partial : H_3(E_K, \partial E_K) \xrightarrow{\sim} H_2(\partial E_K) \cong \mathbb{Z}$

and $0 \to \underset{\substack{\shortparallel \\ \mathbb{Z}}}{H_2(E_K, \partial E_K)} \xrightarrow{\partial} \underset{\substack{\shortparallel \\ \mathbb{Z}^2}}{H_1(\partial E_K)} \xrightarrow{i_*} \underset{\substack{\shortparallel \\ \mathbb{Z}}}{H_1(E_K)} \to 0 \qquad (+)$

previous calc.

Consider $[S', \partial S'] \in H_2(E_K, \partial E_K)$. Let $\ell = \partial[S', \partial S'] = [\partial S']$. By exactness, $\ell \in \ker i_*$. Also ℓ is primitive, since it is represented by the embedded curve $\partial S'$. Consider $j^*: H_1(\partial E_K) \to H_1(U(K))$. Then $j^*[\partial S'] = [K]$ generates $H_1(U(K)) \Rightarrow \ell \neq 0$ in $H_1(\partial E_K)$.

1) follows from $\ell \in \ker i_* \cong \mathbb{Z}$ is a nonzero primitive element
2) follows from 1, since sequence splits $H_1(\partial E_K) = \langle m \rangle \oplus \ker i_*$, and
3) follows from (†), since $\ell = \partial([S', \partial S'])$ generates $\operatorname{im} \partial = \ker i_*$.

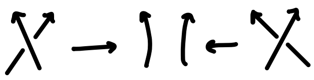
$\square$

Theorem: every $K \hookrightarrow S^3$ has a Seifert Surface

2 proofs:

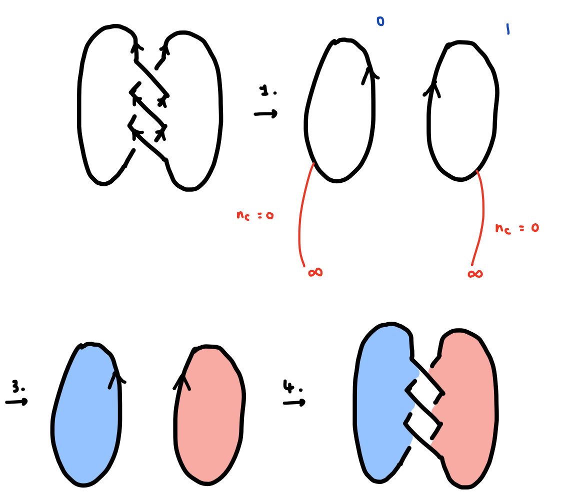
Seifert's algorithm: given a diagram D of K, construct a seifert surface.

1. Give every crossing the oriented resolution:



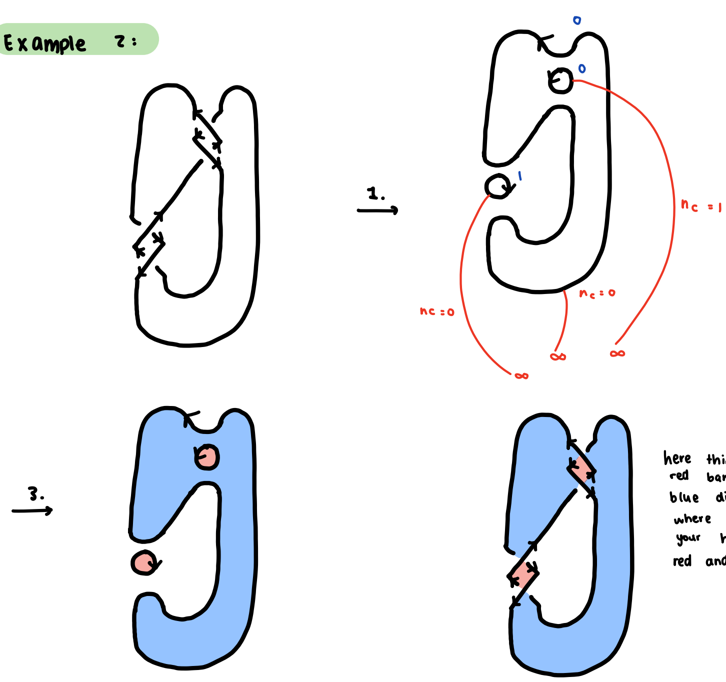
2. Resulting diagram has no crossings, and a natural orientation coming from the orientation on D.
3. Let c be a circle of the resulting diagram. It bounds a disk $D_c \subset \mathbb{R}^2$. Let n_c = mod 2 # of circles in the resolved diagram that separate c from ∞. Let $r_c = 0$ if c is oriented counter clockwise, and $r_c = 1$ if c is oriented clockwise.
   Orient D_c according to sum $n_c + r_c$: Color D_c red if $n_c + r_c$ is odd, and blue if $n_c + r_c$ is even.
4. Attach a band  of surface at each crossing.

Example 1:



rem: orientable

Example 2:

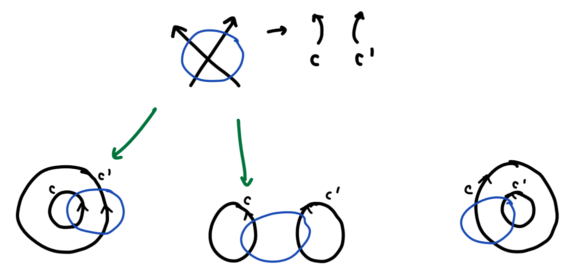


here think about the red band as sitting above blue disc (like a hole where you can put your hand through between red and blue disc).

Proof 1 : Seifert's Algorithm

2. each circle bounds a disk D_c at height h_c above blackboard.

Need to check that : 4) is compatible with orientations. Equivalently, if c, c' are 2 circles at a crossing, then $n_c + r_c \not\equiv n_{c'} + r_{c'}$ (*) mod 2. Resolve all crossings except this one. We get three possible pictures:
(not the same colour)

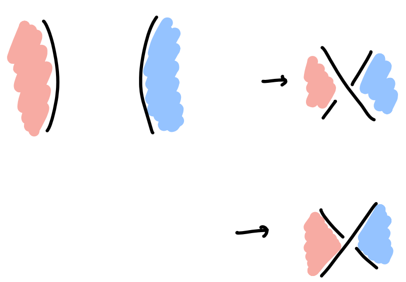


$r_c = r_{c'} = 0$ $\quad$ $r_c \neq r_{c'}$ $\quad$ $r_c = r_{c'} = 1$

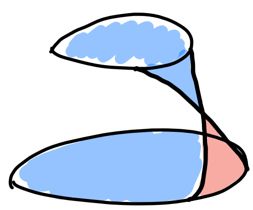
$n_c = n_{c'} + 1$ $\quad$ $n_c = n_{c'}$ $\quad$ $n_c = n_{c'} + 1$

So (*) holds in all three cases. To see that S is embedded, it's enough to look at a neighbourhood of crossing. We have two local models:

$n_c = n_{c'}$, $r_c \neq r_{c'}$.



$n_c = n_{c'} \pm 1$, then $r_c \equiv r_{c'}$ and



* $\partial S \subseteq \partial E_K$ since S is a regular submanifold, so argue by going local and proving it on trivial chart manifold ($\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$).

Proof 2 : (sketch) .

$H^1(E_K; \mathbb{Z}) \cong \mathbb{Z}$ by UCT, say $= \langle a \rangle$. But $H^1(E_K;\mathbb{Z}) \subset H^1(E_K; \mathbb{R}) \overset{UCT}{\cong} \mathbb{R}$, and $H^1(E_K; \mathbb{R}) \cong H^1_{dR}(E_K)$.
Pick $\alpha \in \Omega^1(E_K)$ with $d\alpha = 0$ with $[\alpha] = a \in H^1(E_K; \mathbb{R})$.

Fix some $p_0 \in E_K$, and define $f_\alpha : E_K \to S^1 \cong \mathbb{R}/\mathbb{Z}$. by $f_\alpha(p) = \int_{\gamma_p} \alpha$ where γ_p is a path from p_0 to p.
If γ_p' is another such path,

$$\int_{\gamma_p} \alpha - \int_{\gamma'_p} \alpha = \int_{\gamma_p - \gamma_p'} \alpha = \langle a, [\gamma_p - \gamma_p'] \rangle \in \mathbb{Z}$$ since came from an integral class $a \in H^1(M; \mathbb{Z})$

So $f_\alpha(p)$ is well defined in $\mathbb{R}/\mathbb{Z}$. f_α is a smooth map, so pick $x \in S^1$ a regular value of f_α (Sard's thm).
Then $S = f_\alpha^{-1}(x)$ is a smooth submanifold of E_K with $\partial S \subset \partial E_K$.* We have $[\partial S] = PD(\iota^* a) \in H_1(\partial E_K)$ (exercise). This class is primitive in $H_1(\partial E_K)$ $\Rightarrow$ $[S, \partial S]$ generates $H_2(E_K, \partial E_K) = \mathbb{Z}$
$\Rightarrow$ S is a Seifert surface.

Summary:

(1) Every $K \hookrightarrow S^3$ has a seifert surface, but it's not necessarily unique

(2) The class $[\partial S] \in H_1(E_K)$ generates $\ker\big(H_1(\partial E_K) \to H_1(E_K)\big)$ and satisfies $j_*[\partial S] = [K]$, where $j : \partial E_K \hookrightarrow V(K)$ is inclusion. This does not depend on choice of S.

Definition: $\ell = [\partial S]$ is the homological longitude (seifert longitude) of K.

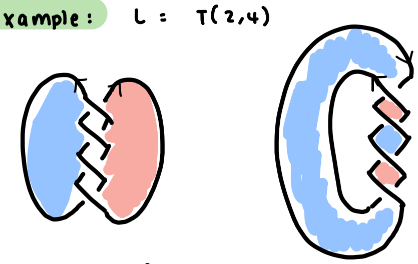
(3) ∂E_K has a preferred basis $\langle m, \ell \rangle$, where m is a meridian for K.

Links: if $L = \coprod^n L_i \hookrightarrow S^3$ is an oriented link where L_i are the components of L, then $\partial E_i = \coprod^n \partial_i E_L$, $\partial_i E_L := \partial(V(L_i))$. So $H_1(\partial_i E_L)$ has a preferred basis $\langle m_i, \ell_i \rangle$, where ℓ_i = Seifert longitude of L_i. (forget other components of L).

Then $\langle m_1, \dots, m_n \rangle$ is a basis of $H_1(E_L)$. But usually $[\ell_i] \neq 0$ in $H_1(E_L)$.

A Seifert surface of L is an embedded oriented $S \hookrightarrow S^3$ with $\partial S = L$. They exist by Seifert's algorithm, but n.b. $\partial_i S \neq \ell_i$.

Example: L : $T(2,4)$

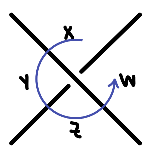


$\chi = 2 - 4 = -2$ $\qquad$ $\chi = 4 - 4 = 0$

2.3 $\pi_1(E_L)$

2 presentations of $\pi_1(E_L)$ coming from a diagram D of L.

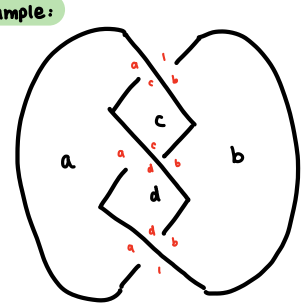
Dehn presentation: Generators are finite regions of $\mathbb{R}^2 \setminus D_{graph}$. Relations $\leftrightarrow$ crossings :



$\longleftrightarrow$ $xy^{-1}zw^{-1} = 1$

infinite region corresponds to $1 \in \pi_1(E_L)$

Example:



generators : a, b, c, d

relations : $1a^{-1}cb^{-1} = 1$

$ca^{-1}db^{-1} = 1$

$da^{-1}1b^{-1} = 1$

presentation : $\langle a, b, c, d \ : \ a^{-1}cb^{-1},\ a^{-1}db^{-1},\ da^{-1}b^{-1} \rangle$

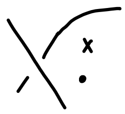
$a^{-1}cb^{-1}$, $a^{-1}db^{-1}$ $da^{-1}b^{-1}$

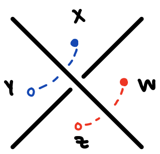
But $c = ab$, $d = ba$

$\Leftrightarrow$ $\langle a, b : aba^{-1}bab^{-1} \rangle$

Take basepoint $* = \infty \in S^3 = \mathbb{R}^3 \cup \{\infty\}$.

generator associated to a region X is a vertical line parallel to z axis passing through X (denote $\bullet$)





$xy^{-1} = wz^{-1}$

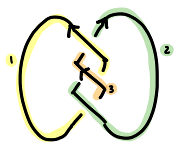
$\bullet$ going into page

$\circ$ going out of page

red and blue are homotopic

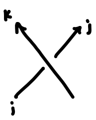
Wirtinger Presentation:

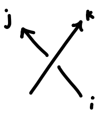
Let D be an oriented planar diagram. An arc of D is part of D that I can draw without lifting up the chalk.

E.g. 

has three arcs given by 3 colors

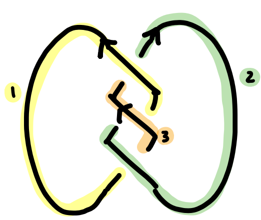
If there are n crossings, then there are n arcs. The group G_{wirt} has generators $\gamma_1, \dots, \gamma_n \leftrightarrow$ arcs and relations $w_1, \dots, w_n$ corresp. to crossings.





$\gamma_j = \gamma_k \gamma_i \gamma_k^{-1}$ $\gamma_j = \gamma_k^{-1} \gamma_i \gamma_k$

Note: all γ_i's are conjugate to each other.

E.g. 

$$\left.\begin{array}{ll} \gamma_2 = \gamma_1 \gamma_3 \gamma_1^{-1} & (+) \\ \gamma_1 = \gamma_3 \gamma_2 \gamma_3^{-1} & (++) \\ \gamma_3 = \gamma_2 \gamma_1 \gamma_2^{-1} & (+++) \end{array}\right\}$$

eliminate γ_3, then

$(++) \Rightarrow \gamma_1 = \gamma_2 \gamma_1 \gamma_2^{-1} \gamma_2 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$

$(+) \Rightarrow \gamma_2 = \gamma_1 \gamma_2 \gamma_1 \gamma_2^{-1} \gamma_1^{-1}$

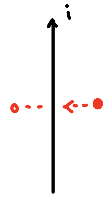
These two relations say the same thing, and

$\gamma_1 = \gamma_2 \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$

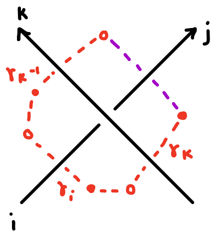
$\gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2$

$$G_{wirt} = \langle \gamma_1, \gamma_2 : \gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2 \rangle$$

Geometry of Picture :



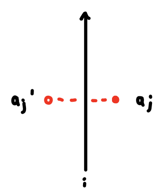
Then γ_i is a loop starting from ∞ and going around arc i compatible with right hand rule.
The relation :



$\gamma_k \gamma_i \gamma_k^{-1}$
net effect $\simeq \gamma_j$

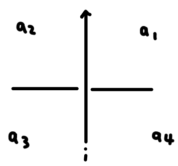
Claim: $G_{Wirt} \cong G_{Dehn}$.

pf: Define a homomorphism $\varphi: G_{Wirt} \to G_{Dehn}$; if we have



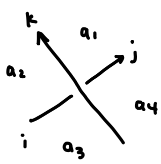
, then let $\varphi(\gamma_i) = a_j (a_j')^{-1}$

This is well defined : if we have



then the Dehn relation says $\gamma(\gamma_i) = a_1 a_2^{-1} = a_4 a_3^{-1}$, which is compatible with our definition

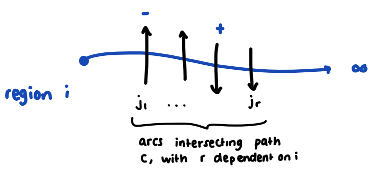
If we have



then

$$\varphi(\gamma_k)\varphi(\gamma_i)\varphi(\gamma_k^{-1}) = (a_4 a_3^{-1})(a_3 a_2^{-1})(a_2 a_1^{-1})$$
$$= a_4 a_1^{-1}$$
$$= \varphi(\gamma_j)$$

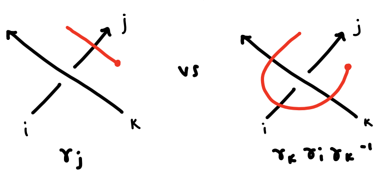
Now define $\psi: G_{Dehn} \to G_{Wirt}$ as follows. Pick a path c_i from region i to infinite region



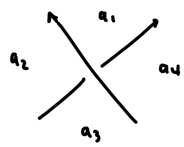
Define $\psi(a_i) = \gamma_{j_1}^{\pm 1} \gamma_{j_2}^{\pm 1} \cdots \gamma_{j_r}^{\pm 1}$, where the exponents are determined by the sign of intersection

E.g. in example, $\psi(a_i) = \gamma_{j_1} \gamma_{j_2} \gamma_{j_3}^{-1} \gamma_{j_4}^{-1}$ need to check signs.

We need to check that $\psi(a_i)$ does not depend on choice of path c_i to ∞. But this is given by the Wirtinger relation.
for example,



Changing the path does not change definition of Ψ. We can check the Dehn relation at a crossing



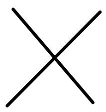
$\Psi(a_1 a^{-1}) = \tau_K = \Psi(a_4 a_3^{-1})$

Exercise: Check φ and Ψ are inverse maps.

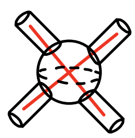
Proof that $\pi_1(E_L, \infty) \cong G_{Dehn}$:

Step 1: let $V(D_{graph})$ be a union of balls around vertices, and $D^2 \times e$ around edge e

e.g.



D_{graph}

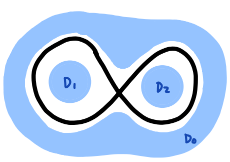


$V(D_{graph})$

Remember our diagram lies on S^2

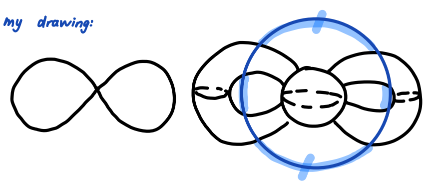
Define $E_D = S^3 \setminus Int(V(D_{graph}))$. Then $E_D \cap S^2$, S^2: plane of the diagram, $= D_0 \cup \dots \cup D_{n+1}$ a union of discs, one for each region, and $D_0 \leftrightarrow$ infinite region.

E.g.



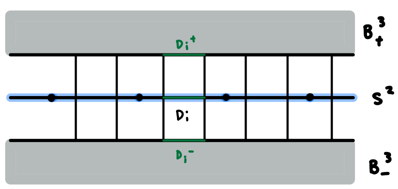
D_{graph}

my drawing:



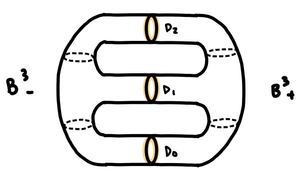
Consider $E_D \cap (S^3 \setminus V(S^2)) = B^3_+ \cup B^3_-$

Picture:



This is where the diagram lies

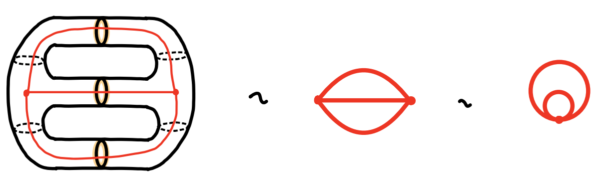
So $E_D \cong B_+^3 \cup_{D_i^+ \sim D_i^-} B^3_-$ is a handlebody which looks like



remember this is solid!

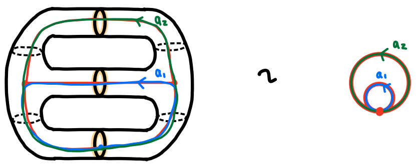
So $E_D \sim \bigvee_{i=1}^{n} S^1$

e.g. the handlebody deformation retracts onto this red graph, which then deformation retracts to a bouquet of n circles.



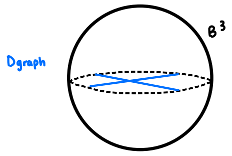
(squash e_0)

Hence $\pi_1(E_D) \cong \langle a_1, \dots, a_{n+1} : \ \rangle$ is a free group with generators a_i, where the a_i are loops on the graph:

e.g. 

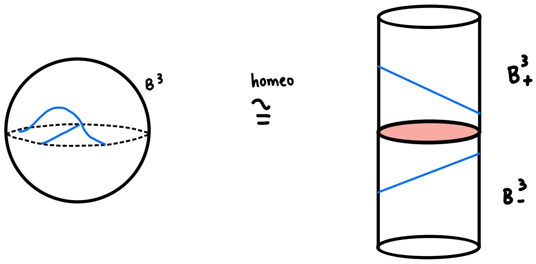
But these a_i are exactly the Dehn generators.

Step 2: look near a crossing.



Note that $B^3 \setminus (B^3 \cap \text{Dgraph})$ strong deformation retracts to $S^2 \setminus (S^2 \cap \text{Dgraph})$ (radially project from origin (= centre of crossing))

Now consider a region of the knot:

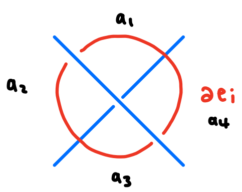


Then $B_\pm \setminus (B_\pm \cap K)$ s.d.r to $\partial B_\pm \setminus (\partial B_\pm \cap K)$

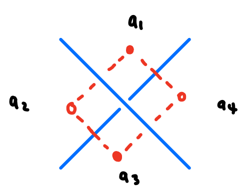
So $B^3 \setminus (B^3 \cap K)$ s.d.r to $S^2 \setminus (S^2 \cap K) \cup 2$ cell ← the red region

So $E_L \sim E_D \cup \{n \text{ 2-cells, one for each crossing}\}$

$\Rightarrow \pi_1(E_L) \cong \pi_1(E_D) / \langle \partial e_i \rangle$, where e_i is the ith 2 cell. What does this boundary look like? Looking down into the can:



The Dehn relator is $a_1 a_2^{-1} a_3 a_4^{-1}$, which geometrically looks like which is exactly



15/02

Ambient Isotopy:

Definition: Suppose $N_0, N_1 \subset M$ are smooth submanifolds. We say N_0 and N_1 are **ambient isotopic** if theres a diffeomorphism $f: M \to M$ such that

1) $f(N_0) = N_1$
2) $f \sim_i id_M$

$\Rightarrow f: M \setminus N_0 \xrightarrow{\sim} M \setminus N_1$

Ambient isotopy $\Rightarrow$ isotopy:

Let $i_0 : N_0 \hookrightarrow M$ be the inclusion, and $i_1 : N_0 \hookrightarrow M$; $i_1 := f \circ i_0$. Then $i_0(N_0) = N_0$, $i_1(N_0) = N_1$ and $f \sim_i id_M$, so $f \circ i_0 \sim_i i_0$, i.e. $i_1 \sim_i i_0$.

Theorem: if N is a compact manifold and $i_0, i_1 : N \hookrightarrow M$ are isotopic, then $N_0 = i_0(N)$ is ambient isotopic to $N_1 = i_1(N)$.

i.e. isotopy (of a compact manifold) $\Rightarrow$ ambient isotopy.

Lemma: Suppose $N \subset M$ is a smooth submanifold, and $v \in \Gamma(TN)$ is a vector field on N. Then $\exists$ $V \in \Gamma(TM)$ with $V|_N = v$.

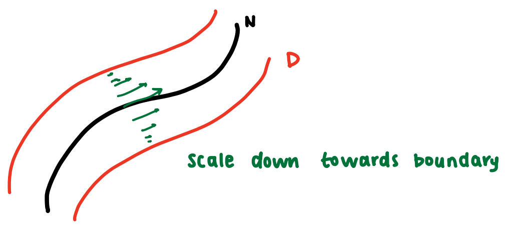
$\|\cdot\|$ = Riemannian metric chosen to define disc bundle on $\nu_{N/M}$

proof: Let $j : D := D(\nu_{N/M}) \hookrightarrow M$ be a tubular neighbourhood of N. Choose a splitting $TD = \pi^*(TN) \oplus \pi^*(\nu_{N/M})$, where $\pi : D \to N$ is the projection. Define $\hat{v} \in \Gamma(TD)$ by $\hat{v}|_w = \rho(\|w\|)\, \pi^*(v|_{\pi(w)}) \in \pi^*(TN) \subset \pi^*(TD)$,

where $\rho : [0,1] \to [0,1]$, $\rho(x) = 0$ if $x \geq 3/4$, $= 1$ if $x \leq 1/4$

Now define $V|_x := dj(\hat{v}|_w)$ if $x = j(w)$, and 0 $x \notin Im(j)$.

Picture :



Proof of Theorem: Suppose $F : N \times I \to N$ is the isotopy, then $\hat{F} : N \times I \to M \times I$ is an embedding, $(x,t) \mapsto (F(x,t), t)$.

Let $\hat{N} = im \hat{F}$. Consider a vector field $v \in \Gamma(T\hat{N})$, $v = d\hat{F}(\frac{\partial}{\partial t})$, i.e. $v(F(x,t),t) = (\frac{dF}{dt}|_{(x,t)}, 1) \in TM \oplus TI = T(M \times I)$

By lemma, v extends to $V \in T(M \times I)$, $V|_{(p,t)} = (V_0(p,t), f(p,t)) \in TM \oplus TI$, where $V_0(p,t)$ is a time-dependent vector field on M. So let $\Phi : M \times I \to M$ be the flow of V, so $\frac{d\Phi}{dt}|_{(p,t)} = V_0(p,t)$ and $\frac{d\Phi}{dt}|_{(F(x,t),t)} = \frac{dF}{dt}|_{(F(x,t),t)}$.

By uniqueness of solutions to ODE's, $\Phi|_{N \times I} = F$, so Φ is an ambient isotopy between $N_0 = F(N,0)$ and $N_1 = F(N,1)$.

$\square$

Corollary: If $i_0, i_1 : N \hookrightarrow M$ are isotopic, and $N_k := im(j_k)$; $j_k : D(\nu_{M/N_k}) \hookrightarrow M$ is a tubular neighbourhood, then N_0 is ambient isotopic to N_1

proof: N_0 and N_1 are ambient isotopic via $f : M \xrightarrow{\sim} M$. $\Rightarrow$ $j_0 \sim f \circ j_0$ is a tubular neighbourhood. By uniqueness of tubular neighbourhood $\Rightarrow$ $f \circ j_0 \sim j_1$ $\Rightarrow$ $j_0 \sim j_1$ $\Rightarrow$ image of j_0 is ambient isotopic to j_1

$\square$

Corollary: if $L_0, L_1 : \hat{\sqcup} S^1 \hookrightarrow S^3$ are isotopic and j_0, j_1 are tubular nhoods of L_0, L_1, then

$$S^3 \setminus im(j_0) \cong S^3 \setminus im(j_1)$$

up to orientation preserving diffeomorphism.

$\square$

Corollary: unknot $\not\simeq$ $T(2,3)$ (trefoil)

proof: $E_u \simeq S^1 \times D^2$, $\pi_1(E_u) = \mathbb{Z}$, and $\pi_1(E_{T(2,3)}) = \langle \gamma_1, \gamma_2 \mid \gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2 \rangle$. This is nonabelian since it has a surjective map $\pi_1(E_{T(2,3)}) \to S_3$; $\gamma_1 \mapsto (1\ 2)$, $\gamma_2 \mapsto (2\ 3)$. So $\pi_1(E_{T(2,3)}) \not\cong \mathbb{Z} = \pi_1(U)$ $\square$

Remark: $\pi_1(E_{T(2,3)}) = \langle x, y \mid x^2 = y^3 \rangle$

2.4) Alexander Polynomial

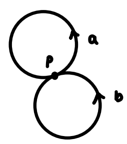
Let $K \subset S^3$ be a knot, and consider the abelianization map $|\cdot| : \pi_1(E_K) \to H_1(E_K) \cong \mathbb{Z}$. Then $\ker|\cdot| \leq \pi_1(E_K)$. By the correspondence between covering spaces and subgroups, there's a connected covering space $p: \tilde{E}_K \to E_K$ with $\pi_1(\tilde{E}_K) = \ker|\cdot|$. But $\ker|\cdot|$ is a normal subgroup, so $\tilde{E}_K$ is a normal covering with deck group $G_{Deck} = \pi_1(E_K)/\ker|\cdot| \cong H_1(E_K) \cong \mathbb{Z}$.

Definition: $\tilde{E}_K$ is the infinite cyclic cover of E_K.

Fact: $E_{T(2,3)} \sim X$, a cell complex with 1 0-cell p, 2 1-cells a, b, and 1 2-cell attached along $w = abab^{-1}a^{-1}b^{-1}$

exterior of $T(2,3)$

$\sim$ means homotopic



So $\pi_1(X) = \langle a, b \mid w \rangle = \pi_1(E_T)$.

We have that $C_*^{cell}(X) : \underset{\langle w \rangle}{\mathbb{Z}} \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \underset{\langle a,b \rangle}{\mathbb{Z}} \xrightarrow{[0,0]} \underset{\langle p \rangle}{\mathbb{Z}}$

$da = p - p = 0$
$da = p - p = 0$

$dw = a + b + a - b - a - b = a - b$

If $e: D^k \to X$ is a cell, then $\pi_1(D^k) = 1$, so e lifts to a map $\tilde{X} \sim \tilde{E}_K$

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ e: D^k \arrow[ur, "\tilde{e}"] \arrow[r] & X \end{tikzcd}$$

and $G_{Deck} = \mathbb{Z} = \langle t^k \rangle$ acts freely and transitively on set of lifts

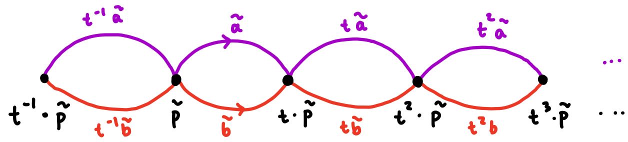
Since D^k is simply-connected, and E_K is path connected, then there always exists such a lift.

$t^{-1}\cdot\tilde{p}$ $\quad\tilde{p}\quad$ $t\cdot\tilde{p}$ $\quad t^2\cdot\tilde{p}$ $\ldots$

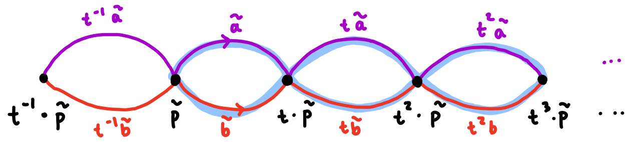
Let $\tilde{a}$ be the lift of a with $\tilde{a}(0) = p$. Then $\tilde{a}(1) = t^{|a|}\cdot\tilde{p} = tp$. Similarly $\tilde{b}(0) = \tilde{p}$, $\tilde{b}(1) = t^{|b|}\cdot\tilde{p} = tp$

↳ choose base points. ($|a|$ = abelianization of a)

Picture of action of G_{Deck}



Let $\tilde{w}$ be the lift of w with $\tilde{w}(0) = \tilde{p}$, then $w = abab^{-1}a^{-1}b^{-1} \Rightarrow \tilde{w}(1) = \tilde{p} = p$



Then $C_*^{cell}(\tilde{X})$ is a module over $R = \mathbb{Z}[G_{Deck}] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$

Looks like $C_*^{cell}(\tilde{X}) = \underset{\langle\tilde{w}\rangle}{R} \longrightarrow \underset{\langle\tilde{a},\tilde{b}\rangle}{R \oplus R} \longrightarrow \underset{\langle\tilde{p}\rangle}{R}$

$d\tilde{a} = t\tilde{p} - \tilde{p}$
$= (t-1)\tilde{p}$
$d\tilde{b} = (t-1)\tilde{p}$

$$\begin{aligned} d\tilde{w} &= \tilde{a} + t\tilde{b} + t^2\tilde{a} - t^2\tilde{b} - t\tilde{a} - \tilde{b} \\ &= \tilde{a}(t^2 - t + 1) - \tilde{b}(t^2 - t + 1) \\ &= (\tilde{a} - \tilde{b})(t^2 - t + 1). \end{aligned}$$

17/02

Recall $K \subset S^3$ a knot. Infinite cyclic cover $p: \tilde{E}_K \to E_K$ with Deck group $G_{Deck} \cong \mathbb{Z} \cong H_1(E_K)$, say $\cong \langle \varphi \rangle$, $\varphi: \tilde{E}_K \to \tilde{E}_K$ a diffeo.

Definition: the Alexander module of K is $A(K) = H_1(\tilde{E}_K)$ as a module over $R = \mathbb{Z}[H_1(E_K)] = \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t^{\pm 1}]$ where $t \cdot x = \varphi_*(x)$.

Ex. a) $K = U$, then $E_K = S^1 \times D^2$, so $\tilde{E}_K = \mathbb{R} \times D^2$, $A(K) = H_1(\tilde{E}_K) = 0$

b) $K = T(2,3)$, then $E_K \simeq X$, with

$$C_*^{cell}(\tilde{X}) = R \xrightarrow[d_2]{\begin{bmatrix} t^2-t+1 \\ -t^2+t-1 \end{bmatrix}} R \oplus R \xrightarrow[d_1]{\begin{bmatrix} t-1 & t-1 \end{bmatrix}} R$$

But $\ker d_1 = \langle (1,-1) \rangle$, and $\operatorname{im} d_2 = \langle (t^2-t+1, -t^2+t-1) \rangle \Rightarrow$

$$A(K) = H_1(\tilde{E}_K) = \ker(d_1) / \operatorname{im}(d_2)$$

$$\cong \frac{\langle (1,-1) \rangle}{\langle (t^2-t+1, -t^2+t-1) \rangle}$$

$$\cong R/(t^2-t+1) \neq 0.$$

Hence $E_U \not\cong E_{T(2,3)}$.

Remark: if $\bar{K}$ is the mirror of K, then there is an orientation reversing diffeomorphism that takes $(S^3, K) \to (S^3, \bar{K})$.
$\Rightarrow$ $E_{\bar{K}}$ is orientation - reversing diffeomorphic to E_K. But the above stuff is insensitive to orientation, in particular $\pi_1(E_K)$ and A_K are too. $\Rightarrow$ $\pi_1(E_K) = \pi_1(E_{\bar{K}})$, $A(K) \cong A(\bar{K})$.

But we know $T(2,3) \neq \overline{T(2,3)} = T(-2,3)$. (e.g. different Jones polynomials)

Theorem (Gordon + Leuke): if E_K is orientation preserving homeomorphic to $E_{K'}$, then $K \sim_i K'$. That is to say, knots are determined by their oriented complements.

$\Rightarrow$ $E_{T(2,3)}$ does not have an orientation reversing homeomorphism. If it did, it would be o.p. homeomorphic to $E_{\overline{T(2,3)}}$, so that $T(2,3) \sim_i \overline{T(2,3)}$, a contradiction.

Let $A(K;\mathbb{Q}) := H_1(\tilde{E}_K;\mathbb{Q})$, so $A(K;\mathbb{Q})$ is a module over $R_\mathbb{Q} = \mathbb{Q}[t^{\pm 1}]$. But $R_\mathbb{Q}$ is a P.I.D.
Notice that $E_K \sim X_K$, which is a cell complex with 1 0-cell, n 1-cells, and $n-1$ 2-cells.
This is from the proof that we gave that $G_{Dehn} = \pi_1(E)$ (built some handlebody which is $\sim$ to a wedge of circles and then attach some 2-cells to it).

$\Rightarrow$ $\tilde{E}_K \sim \tilde{X}_K$ which is a cell complex with cells $t^i e$, $i \in \mathbb{Z}$, and e a cell of X.

$$\Rightarrow C_*^{cell}(\tilde{X}) : R^{n-1} \longrightarrow R^n \longrightarrow R$$

is finitely generated over R $\Rightarrow$ $H_*(E_K;\mathbb{Q})$ is finitely generated over R.

Structure Theorem for finitely generated modules over a P.I.D: (like struc. thm. for f.g. a.g)

$$\Rightarrow H_1(\tilde{E}_K;\mathbb{Q}) \cong \underbrace{R_\mathbb{Q}^k}_{\text{free}} \oplus (\underbrace{R_\mathbb{Q}/p_1 \oplus \cdots \oplus R_\mathbb{Q}/p_r}_{\text{Torsion}})$$

Lemma: $H_1(\tilde{E}_K;\mathbb{Q})$ is a Torsion module over $R_\mathbb{Q}$. (no free part)

proof: We can recover $C_*^{cell}(X)$ by setting $t=1$. Algebraically, this says $C_*^{cell}(X_K) = C_*^{cell}(\tilde{X}_K) \otimes_{R_\mathbb{Q}} M_{t-1}$ where $M_{t-1} := R_\mathbb{Q}/(t-1)$.

By UCT,
$$\underset{\substack{\| \\ H_*(E_K;\mathbb{Q}) \cong \mathbb{Q}}}{H_*(X_K;\mathbb{Q})} \cong \underbrace{H_*(\tilde{X}_K;\mathbb{Q}) \otimes M_{t-1}}_{A} \oplus \underbrace{\mathrm{Tor}(H_{*-1}(\tilde{X}_K;\mathbb{Q})}_{B}, M_{t-1})$$

Consider
$$\begin{aligned} H_0(\tilde{X}_K;\mathbb{Q}) &\cong \mathbb{Q} \quad \text{since } \tilde{X}_K \text{ is connected} \\ &\cong R_\mathbb{Q}/(t-1) \leftarrow \varphi \text{ acts by identity} \\ &\cong M_{t-1} \end{aligned}$$

$H_1(E_K) \cong \mathbb{Z}$ so UCT $\Rightarrow$ $H_1(E_K,\mathbb{Q}) \cong \mathbb{Q}$

So when $*=1$, $B \cong \mathrm{Tor}(H_0(\tilde{X}_K;\mathbb{Q}), M_{t-1}) \cong \mathrm{Tor}(M_{t-1}, M_{t-1}) \cong M_{t-1} \cong \mathbb{Q}$. So we have

$$\mathbb{Q} \underset{①}{\cong} H_1(E_K;\mathbb{Q}) \underset{②}{\cong} H_1(X_K;\mathbb{Q}) \underset{③}{\cong} \overbrace{H_1(\tilde{X}_K;\mathbb{Q}) \otimes M_{t-1}}^{A} \oplus \overbrace{\mathrm{Tor}(H_0(\tilde{X}_K;\mathbb{Q}), M_{t-1})}^{B}$$

① Comes from $H_1(E_K;\mathbb{Z}) \cong \mathbb{Z}$, and so by UCT, $\Rightarrow$ $H_1(E_K;\mathbb{Q}) \cong \mathbb{Q}$.
② Since $E_K \cong X_K$, where X_K is our cell complex description
③ From above, UCT.

$$\cong H_1(\tilde{X}_K;\mathbb{Q}) \otimes M_{t-1} \oplus \mathbb{Q}$$

$\Rightarrow$ $A = H_1(\tilde{X}_K;\mathbb{Q}) \otimes M_{t-1} = 0$. $\Rightarrow$ $H_1(\tilde{X}_K;\mathbb{Q})$ has no free part.
Can then extend this result to higher degrees to see that any $H_*(\tilde{X};\mathbb{Q})$ has no free part. $\square$

A consequence: $A(K;\mathbb{Q}) \cong R_{\mathbb{Q}}/p_1 \oplus \dots \oplus R_{\mathbb{Q}}/p_r$ is a Torsion module.

Define the Alexander polynomial

$$\Delta_K(t) = \prod_{i=1}^{r} p_i \in R_{\mathbb{Q}}$$

to be the "order" of $A(K)$. This is well-defined up to multiplication by units in $R_{\mathbb{Q}}$, i.e. up to multiplication by ct^i, where $c \in \mathbb{Q}$, $i \in \mathbb{Z}$.

Example:
- $\Delta_u(t) \sim 1$ (take $R/\langle 1\rangle =$ 0-module)
- $\Delta_{T(2,3)}(t) \sim t^2 - t + 1$

Where $f \sim g$ means $f = ug$ where u is a unit.

2.5 Fibred knots

Definition: a smooth manifold M fibres over S^1 if there's a submersion $f: M \to S^1$ (every point is regular).
$\Rightarrow$ (Ehresman fibration thm) M is a locally trivial fibre bundle over S^1 with fibre $F = f^{-1}(1)$.

Suppose that M is a smooth 3-manifold. In this case, all of the fibres $f^{-1}(p)$ are diffeomorphic to some surface Σ, and there is a diffeomorphism $\varphi: \Sigma \to \Sigma$ called the monodromy, and $M \cong \Sigma \times [0,1]/\sim$, where $(x,1) \sim (\varphi(x), 0)$.

Exercise: M fibres over $S^1 \Rightarrow \chi(M) = 0$
so T^2 fibres over S^1, but Σ_g does not if $g > 1$.

> If M fibres over S^1, then I want to say that $H_1(M) \cong H_1(S^1)$. Since $\chi(S^1) = 1 - 1 = 0$, $\Rightarrow \chi(M) = \chi(S^1) = 0$.

Given M as above, consider $\tilde{M} = \{(x,t) \in M \times \mathbb{R} \mid f(x) = p(t)\}$ ($\tilde{M}$ is a fibre product of $\mathbb{R}$ and M)

$$\begin{tikzcd} \tilde{M} \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{R} \arrow[d, "p"] \\ M \arrow[r, "f"] & S^1 \end{tikzcd}$$

a) The map $\tilde{f}: \tilde{M} \to \mathbb{R}$; $(x,t) \mapsto t$ is a submersion since f is.
b) The map $\hat{p}: \tilde{M} \to M$; $(x,t) \mapsto x$ is a covering map with deck group $\mathbb{Z}$ since p is a covering map with deck group $\mathbb{Z}$.

Then
a) $\Rightarrow \exists$ diffeomorphism $\alpha: F \times \mathbb{R} \to \tilde{M}$ since $\tilde{M}$ is contractible (any fibre bundle over a contractible base $\overset{\mathbb{R}}{\shortparallel}$ is trivial)
b) $\Rightarrow \exists$ a deck transformation $\beta: \tilde{M} \to \tilde{M}$; $(x,t) \mapsto (x, t+1)$.

Composition

$$(x,t) \longmapsto (x,t) \in \tilde{M} \longmapsto (x, t+1) \longmapsto (x, t+1)$$

$$F \longrightarrow F \times 0 \subset F \times \mathbb{R} \xrightarrow{\alpha} \tilde{M} \xrightarrow{\beta} \tilde{M} \xrightarrow{\alpha^{-1}} F \times \mathbb{R}$$

$$F \times \{0\} \longrightarrow F \times \{1\}$$

defines a map $\varphi: F \to F$ which is called the monodromy of fibration

Then $M = \tilde{M}/\alpha = F\times[0,1] / (\varphi(x),1) \sim (x,0)$ φ or φ^{-1}?

Definition: $K \subset S^3$ is fibred if E_K fibres over S^1. If so, I can choose a fibration $f: E_K \to S^1$ such that $F = f^{-1}(1)$ is connected (exercise)

If K is fibred, then we have a covering map $\tilde{E}_K \to E_K$ with deck group $\mathbb{Z}$ by the construction we just did. This must be the infinite cyclic cover, since the only surjective map $\pi_1(E_K) \to \mathbb{Z}$ is the abelianization.

So by a), $\tilde{E}_K \simeq F\times\mathbb{R}$, and so $H_1(\tilde{E}_K) = H_1(F)$

How does t act? $t\cdot x = \beta_*(x)$, i.e. $t: H_1(F) \to H_1(F)$ is given by φ_*. So as a module over $R\mathbb{Q}$, $H_1(\tilde{E}_K) = H_1(F)\otimes R\mathbb{Q} / (t\cdot x = \varphi_*(x))$. In other words, $H_1(\tilde{E}_K) = \mathrm{coker}(\Phi_*: H_1(F)\otimes R\mathbb{Q} \to H_1(F)\otimes R\mathbb{Q})$. $\Phi_* = (tI - \varphi_*)$.

Summary: if K is fibred (i.e. E_K fibres over S^1), we can write $E_K \simeq \Sigma\times[0,1]/\sim$, where $(x,0) \sim (\varphi(x),1)$, and φ is the monodromy as above (here $\Sigma :=$ fibre $= F$). Consider the map $\Phi: \Sigma\times\mathbb{R} \to \Sigma\times\mathbb{R}$ given by $\Phi(x,t) = (\varphi(x), t+1)$. Then Φ generates a free, properly discontinuous action of $\mathbb{Z}$ on $\Sigma\times\mathbb{R}$. The set $\Sigma\times[0,1]$ is a fundamental domain for the action of Φ, so $(\Sigma\times\mathbb{R})/\Phi \simeq (\Sigma\times[0,1])/\sim \simeq E_K$. The quotient map $p: \Sigma\times\mathbb{R} \to E_K$ is a covering map with Deck group $\mathbb{Z}$. The corresponding homomorphism $\pi_1(E_K) \to \mathbb{Z}$ (correspondence for covering maps) must be the abelianization map, since any map factors through $H_1(E_K) \simeq \mathbb{Z}$. Hence p is the cyclic infinite cover, and we have that:

Prop: If E_K fibres over S^1 with monodromy $f: \Sigma \to \Sigma$, then $\tilde{E}_K \simeq \Sigma\times\mathbb{R}$. The action of the deck group is generated by the map $(x,t) \mapsto (\varphi(x), t+1)$.

Corollary: If E_K fibres over S^1 with monodromy $f: \Sigma \to \Sigma$, then

$$\Delta_K(t) \sim \det(tI - \varphi_*)$$

where $\varphi_*: H_1(\Sigma) \to H_1(\Sigma)$ is the homomorphism induced by the monodromy.

proof: For the isomorphism $H_1(\tilde{E}_K) \simeq H_1(\Sigma\times\mathbb{R}) \simeq H_1(\Sigma)$, the map $\Phi_*: H_1(\tilde{E}_K) \to H_1(\tilde{E}_K)$ is given by $\varphi_*: H_1(\Sigma) \to H_1(\Sigma)$. Hence if $e_1, \dots, e_{2g}$ is a basis for $H_1(\Sigma)$ over $\mathbb{Z}$ (remember any orientable surface is homotopic to Σ_g for some g, and $H_1(\Sigma_g) \simeq \mathbb{Z}^{2g}$), the $\mathbb{Z}[t^{\pm 1}]$-module $H_1(\tilde{E}_K)$ will be generated by the e_i, with relations $te_i = \varphi_*(e_i)$. I.e. $(\varphi_* - tI)e_i = 0$. So $H_1(\tilde{E}_K)$ has square presentation matrix of the form $(tI - \varphi_*)$

Corollary: If K is a fibered knot, then $\Delta_K(t)$ is monic and of degree $2g$, where g is the genus of the fibre.

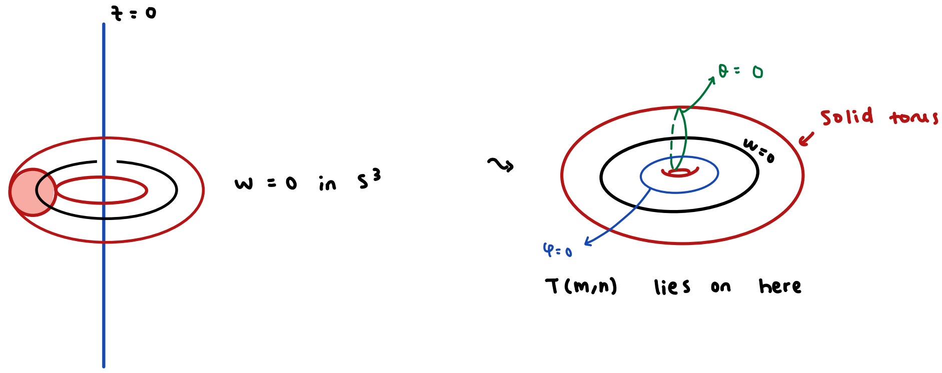
2.6 Torus knots

Consider S^3 as $S^3 \subset \mathbb{C}^2 := \{ (z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1 \}$. Can also identify S^3 with $\mathbb{R}^3$ via stereographic projection from $(0,i)$. This identifies $S^1 \times 0$ with the unit circle in (x,y)-plane in $\mathbb{R}^3$

Define $T(m,n)$ (the (m,n)-torus knot) to be $T(m,n) = \{ (z,w) \in S^3 : z^m = w^n \}$

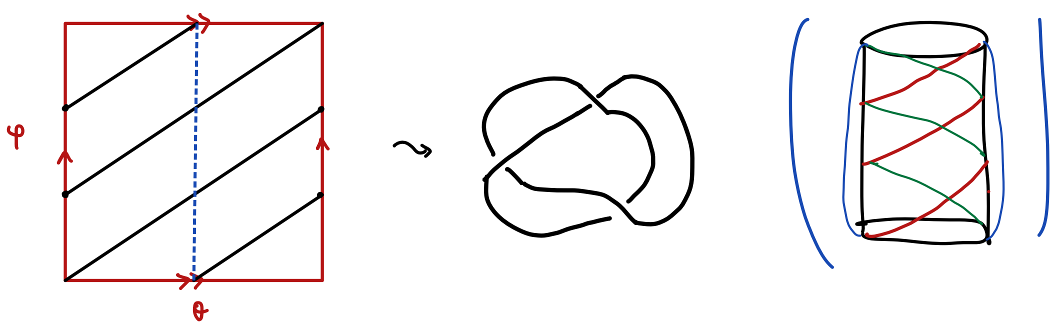
If so, we have $|z|^2 + |w|^2 = 1$ and $|z|^m = |w|^n$, so $|z|^2 + |z|^{2m/n} = 1$. Note $f(r) = r^2 + r^{2m/n}$ is a monotonic increasing function of r, so $\exists!$ r with $|z| = r$ satisfying these equations. So $T(m,n)$ lies on the torus $\{ (z,w) \in S^3 : |z| = r \text{ and } |w|^2 + |z|^2 = 1 \}$

This torus looks like



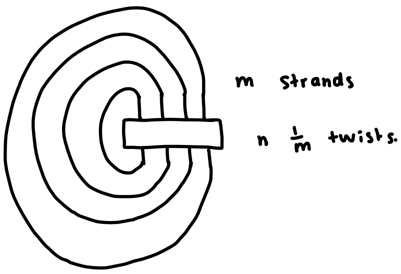
Write $z = re^{2\pi i \theta}$, $w = r'e^{2\pi i \varphi}$, then $m\theta = n\varphi \pmod 1$.

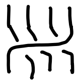
Then $T(m,n)$ is a line with slope m/n, e.g $m = 2$, $n = 3$, then

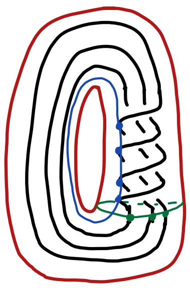


The map $f : E_{T(m,n)} \to S^1$ given by $f(z,w) = \dfrac{z^m - w^n}{|z^m - w^n|}$ is a submersion, so $T(m,n)$ is a fibred knot.

In general, $T(m,n)$ has a diagram



$1/m$ twist: 

For example, $T(3,4)$: 

Consider $f: S^3 \setminus T(m,n) \to \mathbb{C}\setminus\{0\} \to S^1$.

$(z,w) \longmapsto z^n - w^m$

$\xi \longmapsto \frac{\xi}{\|\xi\|}$

This map is a submersion (check!), and $T(m,n)$ is fibred.

Plenty of other knots are fibred, e.g. figure 8 knot.

2.7 Presentations

Definition: Suppose M is a module over a commutative ring R. Then M is finitely presented if there's an exact sequence $0 \to R^n \xrightarrow{P} R^m \xrightarrow{\pi} M \to 0$

and this sequence is a presentation of M.

If $e_i = (0,\dots,1,\dots 0) \in R^m$, then $\pi(e_1),\dots,\pi(e_m)$ are generators, and $P(e_1),\dots,P(e_n)$ are relations between the generators

$\downarrow$
$\pi \circ P = 0$

Write $P(e_j) = \sum_{i=1}^{m} P_{ij} e_i$. Say $m \times n$ matrix $[P_{ij}]$ is a presentation matrix for M. (m rows, n columns.)

Fact: If P and P' are two presentation matrices for M, then they are related by a sequence of elementary operations and their inverses.

Denote: generators $a_i := \pi(e_i)$, $\leftrightarrow$ rows

relations $r_j = P(e_j)$. $\leftrightarrow$ columns

> General idea: $P(e_j) = \sum_{i=1}^{m} P_{ij} e_i$ for some coefficients P_{ij}. Now, applying the linear map π to the 'relation' represented by $P(e_j)$ gives us $\pi(P(e_j)) = \sum_{i=1}^{m} P_{ij}\,\pi(e_i) = \sum_{i=1}^{m} P_{ij} a_i$. Notice that we're summing down the column.

Then the moves are:

Moves:

1) add a new generator a_{m+1} and relation: $a_{m+1} = 0$.

$$P \longleftrightarrow \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} = P'$$

The new matrix has the same generators and relations coming from P, but now we have an added one → one more generator (from one more row) and the corresponding added relation

$$\pi(P'(e_{m+1})) = \pi(0e_1 + 0e_2 + \dots + 1e_{m+1}) = 0$$
$$= \pi(e_{m+1}) = a_{m+1} = 0.$$

2) add a new relation $0 = 0$

replace a_i with $a_i + \alpha a_j$.

then

$$P \longleftrightarrow (P \;\; 0)$$

3) Replace a_i with $a_i + \alpha a_j$ for some $\alpha \in R$

$P \longleftrightarrow P'$, where you do an elementary row operation
j^{th} row of P' is (j^{th} row of P) + $-\alpha$(i^{th} row of P)

Pass through π: if you have $p(e_j) = P_{i1} e_1 + \dots + P_{im} e_m$

The general idea: the relations need to stay the same, in the sense that replacing a_i with $a_i + \alpha a_j$ as a generator does not mean e.g. $a_i + a_1 = 0 \Rightarrow (a_i + \alpha a_j) + a_1 = 0$. Let's think about what happens when we look at the columns of the matrix. We'll use stupid notation, sorry.

Remark that $\pi(p(e_j)) = \pi\left(\sum_{i=1}^{m} P_{ij} e_i\right) = \sum_{i=1}^{m} P_{ij} a_i = P_{1j} a_1 + \dots + P_{m} \cdot a_m = 0$

Replacing say a_k with $a_k + \alpha a_\ell$ gives us for all $j = 1, \dots, n$

$$P_{1j} a_1 + \dots + P_{kj} a_k + \dots + P_{mj} a_m \longrightarrow P_{1j} a_1 + \dots + P_{kj}(a_k + \alpha a_\ell) + \dots + P_{mj} a_m$$

For this to be a relation, we'll need $\forall j$ (columns) to subtract $P_{kj} \alpha a_\ell$. This becomes then

$$P_{1j} a_1 + \dots + P_{kj}(a_k + \alpha a_\ell) + \dots + (P_{\ell j} - P_{kj}\alpha) a_\ell + \dots + P_{mj} a_m = 0.$$

Clearly then the ℓ^{th} row is then (ℓ^{th} row) $-$ α(j^{th} row).

4) Replace r_i with $r_i + \beta r_j$, $\beta \in R$

$P \longleftrightarrow P''$ where i^{th} column of P'' is i^{th} column of P + β j^{th} column of P

Relation $\pi(r_i) = 0$ and $\pi(r_i + \beta r_j) = \pi(r_i) + \beta \pi(r_j) = 0$
This is pretty much immediate

5) Multiply rows + columns by units

universal factorization domain? principal ideal domain

Now suppose that R is a UFD (weaker than PID). If $\alpha_1, \dots, \alpha_k \in R$, then there is a gcd $\gcd(\alpha_1, \dots, \alpha_k)$ that is well defined up to multiplication by a unit.

Definition: If P is an $m \times n$ matrix over a UFD R, let $e_0(P)$ be the $\gcd(\{\det(\tilde{P})\})$, over all $\tilde{P}$ s.t $\tilde{P}$ is an $m \times m$ submatrix obtained by deleting columns of P if $m \le n$ or 0 if $m > n$.

Lemma: If P and P' are related by an elementary move, then $e_0(P) \sim e_0(P')$, where $\sim$ means equal up to multiplication by a unit.

Sketch of proof: Just check for each move using the fact that det is linear on rows and columns, and fact that $\gcd(x, y) \sim \gcd(x, y + \alpha x)$.

Eg: adding rows and columns, may not always be case that you get e.g. both columns i and j in the $m \times m$ submatrix, but there will be another submatrix with the column that was added. Then taking the gcd and using the last observation above, the gcd remains the same.

Definition: If M is a finitely presented module over a UFD R, let $e_0(M) = e_0(P)$, where P is any presentation matrix of M. Then the lemma exactly says that this is well-defined.

Example: If R is a PID, then $e_0(M) = \text{order}(M)$ if M is Torsion, or 0 otherwise.

pf: Choose a presentation matrix in Smith-normal form.

2.8 Multivariable Alexander Polynomial

Suppose $L \subset S^3$ is a link with n components.

Definition: The universal abelian cover $p: \tilde{E}_L \to E_L$ is the connected covering space given by the kernel of the abelianisation map $|\cdot| : \pi_1(E_L) \to H_1(E_L) = \langle m_1, \dots, m_n \rangle \cong \mathbb{Z}^n$.

$\tilde{E}_L$ has $G_{Deck} \cong H_1(E_L) \cong \mathbb{Z}^n$, so $H_1(\tilde{E}_L)$ is a module over $\mathbb{Z}[H_1(E_L)] \cong \mathbb{Z}[\mathbb{Z}^n] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] =: R_L$.

> Recall that the Deck group of the connected covering space corresponding to the subgroup $\operatorname{Ker}|\cdot| \le \pi_1(E_L)$ is given by $\pi_1(E_L)/\operatorname{Ker}|\cdot|$. But this is precisely $H_1(E_L)$ (tagline: $H_1(E_L)$ is the abelianization of $\pi_1(E_L)$

So $\mathbb{Z}[t_1, \dots t_n]$ is a UFD $\Rightarrow$ R_L is a UFD.

Definition: the multivariable Alexander polynomial $\Delta(L) = e_0(H_1(\tilde{E}_L)) \in R_L$, well defined up to multiplication by a unit in R_L : $\pm t_1^{a_1} \cdots t_n^{a_n}$

remember this means we find a presentation for $H_1(\tilde{E}_L)$, and then any presentation matrix,

Example: $H_1(E_{T(2,3)}) \cong R_K / (t^2 - t + 1)$

$$\Rightarrow \Delta(T(2,3)) \sim t^2 - t + 1$$

2.9. Fox Calculus

Suppose X is a cell complex with 1 0-cell P, m 1-cells $a_1, \dots, a_m$, and n 2-cells attached along $w_1 \dots w_m$, words in the a_i's.

Van Kampen $\Rightarrow$ $\pi_1(X) = \langle a_1, \dots, a_m \mid w_1, \dots, w_n \rangle$, and $H_1(X) = \mathbb{Z}^k \oplus T$, T is torsion. Define

$$\overline{H_1(X)} := H_1(X)/_T \cong \mathbb{Z}^k$$

We have the abelianization map giving a homomorphism

$$\begin{tikzcd} 1\cdot 1: \pi_1(X) \arrow[r] \arrow[rr, bend right, "\text{surjective.}"'] & H_1(X) \arrow[r, "q"] & \overline{H_1(X)} \cong \mathbb{Z}^k \end{tikzcd}$$

Let $p: \tilde{X} \to X$ be the covering map corresponding to $\ker 1\cdot 1$. Then $G_{Deck} \cong \mathbb{Z}^k$. Then $\tilde{X}$ will be a cell complex, cells are of the form $g\tilde{e}$, where $g \in G_{Deck}$, and $\hat{e}$ is a lift of a cell e based at $\tilde{p}$ (preferred lift of p (0-cell)).

Since cells in $\tilde{X}$ are lifts of cells in X, the boundary operator in $C_*^{cell}(X)$ commutes with the action of G_{Deck}. So $C_*^{cell}(\tilde{X})$ is a chain complex over $R_x = \mathbb{Z}[\overline{H_1(X)}] \cong \mathbb{Z}[\mathbb{Z}^k] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$.

Then
$$C_*^{cell}(\tilde{X}): \underset{\langle \hat{w}_j \rangle}{R_x^n} \xrightarrow{d_2} \underset{\langle \hat{a}_i \rangle}{R_x^m} \xrightarrow{d_1} \underset{\langle \hat{p} \rangle}{R_x}$$

remember $G_{Deck} \cong \mathbb{Z}^k \cong \overline{H_1(X)}$ which is generated by $|a_i|$

What are these boundary maps?

d_1) $\tilde{a}_i$ starts at $\tilde{p}$ and ends at $|a_i|\hat{p}$

$\Rightarrow$ $d_1(\tilde{a}_i) = (|a_i| - 1)\hat{p}$

So d_1 has matrix $[\,|a_1| - 1 \;\; \dots \;\; |a_m| - 1\,]$ $(1 \times m)$ matrix.

d_2) $d_2: R_x^n \xrightarrow{A_x} R_x^m$, where $A_x = [\,d a_i\, w_j\,]$, where $d a_i\, w$ is the so-called Fox derivative, given by

$$d a_i \left(\prod_{k=1}^{r} a_{i_k}^{\pm 1} \right) = \sum_{k=1}^{r} |a_{i_1}^{\pm} \dots a_{i_{k-1}}^{\pm 1}| \, d a_i (a_{i_k}^{\pm 1}) \quad \in \mathbb{Z}[\overline{H_1(x)}]$$

and
$$d a_i(a_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \qquad d a_i(a_j^{-1}) = \begin{cases} -|a_j^{-1}| & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Proof: $d a_i(\tilde{w}_j)$ counts segments of $\tilde{w}_j$ that run along $g\tilde{a}_i$, $g \in G_{Deck}$. As we walk along $\tilde{w}_j$, segments we pass over correspond to the lifts of the generators in $\tilde{w}$, we first pick out those that run over $g\tilde{a}_i$ $\leftrightarrow$ appearances of a_i in w. Lift of a_i corresponding to $\alpha a_i \beta$ is exactly $|\alpha|\tilde{a}_i$. The lift corresponding to $\alpha a_i^{-1} \beta$ is $-|\alpha||a_i^{-1}|\tilde{a}_i$.

$d_2: R_x^n \xrightarrow{A_x} R_x^m$ n relations, m generators. So $n \times m$ matrix, $[\,d a_i\, w_j\,]_{(i,j) \in [n,m]}$. The poly $d a_i\, w_j$ kind of counts the change along a_i of w_j's boundary in the lift, if that makes any sense. I think it does. Look at the eqn:

$$d a_i \underbrace{\left(\prod_{k=1}^{r} a_{i_k}^{\pm 1} \right)}_{\text{a word } w} = \sum_{k=1}^{r} |a_{i_1}^{\pm} \dots a_{i_{k-1}}^{\pm 1}| \, d a_i (a_{i_k}^{\pm 1})$$

Lemma: $d_{a_i}(ww') = d_{a_i} w + |w| d_{a_i} w'$ (Leibniz rule)

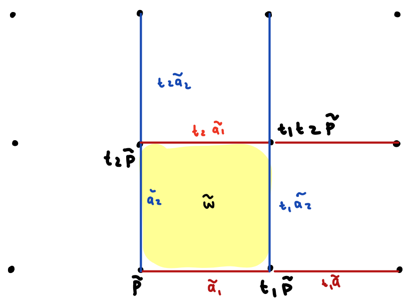
proof: almost follows from definition, but should check $d_{a_i}(\alpha a_i)(a_i^{-1}\beta) = d_{a_i} \alpha\beta$
$= d_{a_i} \alpha (a_i a_i^{-1})\beta$

follows from $d_{a_i} a_i a_i^{-1} = 1 + |a_i|(-|a_i^{-1}|) = 1-1=0$

Example 1: $\pi_1(X) = \langle a_1, a_2 \mid a_1 a_2 a_1^{-1} a_2^{-1} \rangle$
i.e. $X = T^2$, $X \sim E_{T(2,2)}$

Abelianize $\pi_1(X)$: $a_1 + a_2 - a_1 - a_2 = 0$
$\Rightarrow H_1(X) = \langle t_1, t_2 \rangle \simeq \mathbb{Z}^2$, where $|a_i| = t_i$

Then $\tilde{X}$



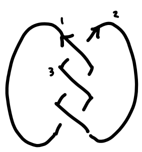
remember $|\cdot|$ is a homomorphism

$d_{a_1} w = d_{a_1}(a_1 a_2 a_1^{-1} a_2^{-1}) = 1 - |a_1 a_2 a_1^{-1}| = 1 - t_2$

$d_{a_2} w = d_{a_2}(a_1 a_2 a_1^{-1} a_2^{-1}) = |a_1| - |a_1 a_2 a_1^{-1} a_2^{-1}| = t_1 - 1$

So $C_*^{cell}: R \xrightarrow{\begin{bmatrix} 1-t_2 \\ t_1-1 \end{bmatrix}} R^2 \xrightarrow{[t_1-1 \;\; t_2-1]} R$, then $d^2=0$ since $[t_1-1 \;\; t_2-1]\begin{bmatrix} 1-t_2 \\ t_1-1 \end{bmatrix} = 0$

Example 2: recall for trefoil



$\pi_1(X) = \langle a_1, a_2, a_3 \mid a_2^{-1} a_1 a_3 a_1^{-1}, a_1^{-1} a_3 a_2 a_3^{-1} \rangle$ (Wirtinger presentation)

Abelianize: $|a_1|, |a_2|, |a_3|$
$-|a_2| + |a_1| + |a_3| - |a_1| = 0$ $\quad |a_2| = |a_3|$
$-|a_1| + |a_3| + |a_2| - |a_3| = 0$ $\quad |a_1| = |a_3|$

Hence $H_1(X) = \mathbb{Z}$ generated by $t = |a_1| = |a_2| = |a_3|$.

Then $A_X = \begin{bmatrix} t^{-1}-1 & -t^{-1} \\ -t^{-1} & 1 \\ 1 & t^{-1}-1 \end{bmatrix}$

Check that $d^2 = 0$

2.10 Group Presentations

Suppose $G = \underbrace{\langle a_1, \dots, a_m \mid w_1, \dots, w_n \rangle}_{P}$ is a finitely presented group

To the presentation P, we associate a 2-complex X_P as before : 1-cells $\leftrightarrow a_i$, 2-cells $\leftrightarrow w_j$.
Let $A_P = A_{X_P}$ be the Alexander matrix

Tietze moves are elementary moves on presentations: preserve isomorphism type of the group

1) add a new generator a_{m+1}, $w_{m+1} = a_{m+1}$

$$P' = \langle a_1, \dots, a_m, a_{m+1} \mid w_1, \dots, w_n, a_{m+1} \rangle$$

$$A_{P'} = \begin{bmatrix} A_P & 0 \\ 0 & 1 \end{bmatrix}$$

2) Add a trivial relation

$$P' = \langle a_1, \dots, a_m \mid w_1, \dots, w_n, \overset{1}{\emptyset} \rangle$$

$$A_{P'} = \begin{bmatrix} A_P & 0 \end{bmatrix}$$

3) Multiply one relation by another

$$P' = \langle a_1, \dots, a_m \mid w_1, \dots, w_i w_j, \dots w_n \rangle$$

Let $w_i' = w_i w_j$ $\quad j \neq i$
$w_j' = w_j w_i$

Note that $d_{a_k} w_i' = d_{a_k} w_i + |w_i| \, d_{a_k} w_j$

$$= d_{a_k} w_i + d_{a_k} w_j$$

So i^{th} col of A_P is the $i^{th} + j^{th}$ col. of A_P

4) Replace w_j by $w_j' = a_i w_j a_i^{-1}$

$$d_{a_k}(w_j') = |a_i| \, d_{a_k}(w_j)$$

So multiply the j^{th} column of A_P by $|a_i|$ (a unit) to get $A_{P'}$.

Theorem (Tietze) if P and P' are presentations of isomorphic groups, then we can get from P to P' by a sequence of Tietze moves.

Definition: if P is a group presentation with m generators and n relations, let

$$\Delta(P) = e_1(A_P) = \gcd\left(\det \tilde{A} \;\middle|\; \tilde{A} \text{ is an } \overset{n \times n}{(m-1) \times (m-1)} \text{ submatrix of } A_P \right)$$

Thm: if P and P' are related by Tietze moves, then $\Delta(P) \sim \Delta(P')$.

idea of proof: check effect of operations 1 to 4 on $e_1(A_P)$ (up to a unit).

So if G is a finitely presented group, define $\Delta(G) = \Delta(P)$ where P is a presentation of G, to be the multivariable Alexander polynomial.

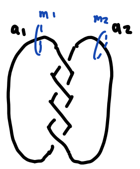
Next time: $\Delta(L) = \Delta(\pi_1(E_L))$.

Example: $G = \pi_1(E_{T(2,3)}) = \langle a_1, a_2, a_3 \mid a_2^{-1}a_1a_3a_1^{-1},\ a_1^{-1}a_3a_2a_3^{-1} \rangle$

Found that $A_P = \begin{bmatrix} t^{-1}-1 & -t^{-1} \\ -t^{-1} & 1 \\ 1 & t^{-1}-1 \end{bmatrix}$

check that all 3 determinants (up to x by unit) are $t^2 - t + 1 \sim \Delta(T(2,3))$

Example: take $L = T(2,4)$



Then can show (example sheet) $\pi_1(E_L) = \langle a_1, a_2 \mid a_1a_2a_1a_2a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1}$
(find wirt. presentation and simplify)

Abelianize: $[\,|a_1|, |a_2| \mid |a_1| + |a_2| + |a_1| + |a_2| - |a_1| - |a_2| - |a_1| - |a_2|\,] = \langle |a_1|, |a_2| \rangle = \langle t_1, t_2 \rangle$
with $m_1 \to t_1$, $m_2 \to t_2$ $= H_1(E_L) \ni \langle m_1, m_2 \rangle$

Compute $A_P = \begin{bmatrix} d_{a_1} w \\ d_{a_2} w \end{bmatrix} = \begin{bmatrix} 1 + t_1t_2 - t_1t_2^2 - t_2 \\ t_1 + t_1^2t_2 - t_1t_2 - 1 \end{bmatrix} = \begin{bmatrix} (1+t_1t_2)(1-t_2) \\ (1+t_1t_2)(t_1-1) \end{bmatrix}$

So $\gcd = 1 + t_1t_2$, $\Delta(L) \sim 1 + t_1t_2$

Note: $d^2 = 0$ says exactly $(t_1 - 1,\ t_2 - 1)\begin{pmatrix} d_{a_1} w \\ d_{a_2} w \end{pmatrix} = 0$

Now suppose P has one more generator than relator (i.e. $n = m-1$)
(e.g. $P = G_{\text{Dehn}}$, or $P = G_{\text{wirt}}$ (if you multiply all the relations together you get 1, so dependent, so can toss one relation).

Then $\Delta(G) = \gcd(\det A_{P,\hat{i}})$, where $A_{P,\hat{i}}$ is A_P with the i^{th} row deleted.

Proposition: $(|a_j| - 1)\det A_{P,\hat{i}} \sim (|a_i| - 1)\det A_{P,\hat{j}}$.

proof: $A_P = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$

Since $d^2 = 0$ in $C_*^{\text{cell}}(\tilde{X}_P)$, then $\sum_i (|a_i| - 1)v_i = 0$, since $d_1 = [\,|a_1| - 1 \ \cdots \ |a_n| - 1\,]$. we just compute

$$(|a_j|-1)\det A_{p,\hat{i}} = \det\begin{bmatrix} v_1 \\ \vdots \\ \hline v_i \\ \vdots \\ (|a_i|-1)v_j \\ \vdots \\ v_m \end{bmatrix} = \det\begin{bmatrix} v_1 \\ \vdots \\ \hline v_i \\ \vdots \\ -\sum(|a_k|-1)v_k \\ \vdots \\ v_m \end{bmatrix} = \det\begin{bmatrix} v_1 \\ \vdots \\ \hline v_i \\ \vdots \\ -(|a_i|-1)v_i \\ \vdots \\ v_m \end{bmatrix} \sim \det(|a_i|-1)\det A_{p,\hat{j}}.$$

using remark from above

det is linear in rows

$\square$

Equivalently, $\dfrac{\det A_{p,\hat{i}}}{|a_i|-1} \sim \dfrac{\det A_{p,\hat{j}}}{|a_j|-1} \implies (|a_i|-1)\overbrace{\gcd(\det A_{p,\hat{k}})}^{\Delta(G)} = \alpha \det A_{p,\hat{i}}$

where $\alpha = \gcd(|a_1|-1,\dots,|a_m|-1)$.

Exercise: $\alpha = \begin{cases} t-1 & \text{if } H_1(X_p)\cong\mathbb{Z} \text{ (one variable poly. ring) } (|L|=1) \\ 1 & \text{if } H_1(X_p)\cong\mathbb{Z}^k \quad k>1 \text{ (more than one component link)} \end{cases}$

E.g. if $(a_1,\dots,a_m)$ generate $\mathbb{Z}$, then $\gcd(t^{a_1}-1,\dots,t^{a_m}-1)=t-1$ (case 1)

But $\gcd(t_1-1,t_2-1)=1$ (case 2)

Corollary: $(|a_i|-1)\Delta(G) \sim \alpha\det A_{p,\hat{i}}$.

So to compute the alexander polynomial, dont need to look at <u>all</u> the determinants, just need to look at one of them, and divide by right factor.

Proposition: Suppose K is a knot. Then $\Delta(E_K)\sim e_0(H_1(\tilde{E}_K))$.

proof: Use $P = G_{\text{wirt}}$. All a_i's are conjugate, so $|a_i| = |a_j| = t$. Hence $d_0 = [t-1 \;\cdots\; t-1]$, and $\ker d_0 = \{(x_1,\dots,x_m) : \sum x_i = 0\}$.

think he means d_1

Consider the map $\pi_{\hat{m}} : \ker d_1 \xrightarrow{\sim} R^{m-1}$: projection on first $m-1$ coords. So

$$H_1(\tilde{E}_K) = \frac{\ker d_1}{\operatorname{Im} d_2} \xrightarrow{\pi} \frac{R^{m-1}}{\operatorname{im}(\pi_{\hat{m}}\circ A_p)} = \frac{R^{m-1}}{\operatorname{im} A_{p,\hat{m}}}$$

$\Rightarrow e_0(H_1(\tilde{E}_K)) = \det(A_{p,\hat{m}}) \sim \Delta(E_K).$

$\square$

2.11 · Seifert Genus

Recall if $K \hookrightarrow S^3$, a Seifert surface of K is a compact, connected, oriented surface $S \hookrightarrow S^3$ with $\partial S = K$.

Dfn: if $K \hookrightarrow S^3$ a knot, it's seifert genus $g(K) = \min \{ g(S) \mid S \text{ is a seifert surface of } K \}$

Proposition : $g(K) = 0 \Leftrightarrow K = \mathcal{U}$

proof: $g(\mathcal{U}) = 0$ obviously.

the disk

is a knot

if $g(K) = 0$, let $\varphi : B^2 \hookrightarrow S^3$ be a genus 0 seifert surface. For $t \in [0,1]$, let $K_t = \varphi|_{\partial B_t}$, where B_t is the ball of radius t. Then K_t is a knot in S^3 with $K_t \sim_i K_1 = K$ via the isotopy $\varphi|_{B^2 \setminus \mathring{B}_t}$.

For small ε, $\varphi|_{B_\varepsilon} \sim d\varphi|_0|_{B_\varepsilon}$. And $\operatorname{im}(d\varphi|_0) \subset$ a plane, so $K_\varepsilon \sim_i K' \subset$ a plane $\Rightarrow K = \mathcal{U}$.

Idea: if a knot is contained in the plane, then it must be the unknot. K_ε is isotopic to a knot contained in a plane, which must be the unknot by our remark.

$\tilde{E}_K$ via Seifert surfaces

Let S be a Seifert surface of K

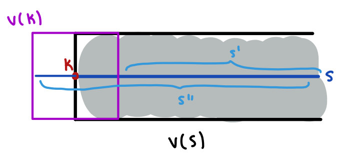
Lemma : $\nu_{S^3/S}$ is trivial.

proof: a real line bundle ($\dim \nu_{S^3/S} = 3-2=1$) $\begin{smallmatrix} \mathbb{R} \to L \\ \downarrow \\ B \end{smallmatrix}$ is trivial iff L is orientable. Now S^3 is orientable, and S is orientable $\Rightarrow \nu_{S^3/S}$ is orientable. So it's trivial. $\square$

by assumption of section

Let $\nu(S)$ be a closed tubular nhood of S. By our lemma, $\nu(S) \overset{\text{homeo.}}{\cong} S \times [-1,1]$.

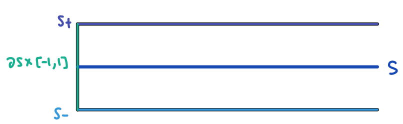
Cross sectional view :

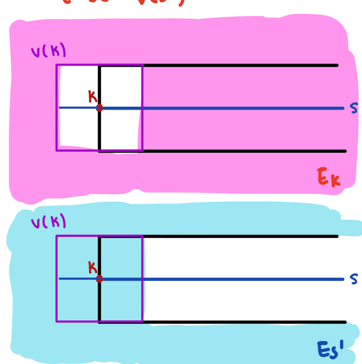


Then $S \sim_i S' \sim S''$, so if $E_S = S^3 \setminus \operatorname{int}(\nu(S))$, then $E_S \overset{\text{homeo}}{\cong} E_{S'} \cong E_{S''}$. $\Rightarrow E_S \cong E_K \setminus E_{S'}$

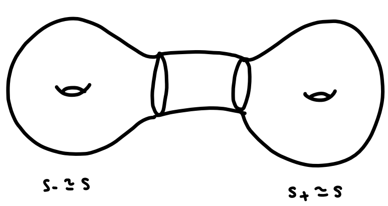
maybe supposed to be $\nu(S'')$

Then $\partial E_S = \partial \nu(S) = \underset{S_-}{S \times (-1)} \cup_{\partial S \times \{-1\}} \partial S \times [-1,1] \cup_{\partial S \times \{1\}} \underset{S_+}{S \times 1}$



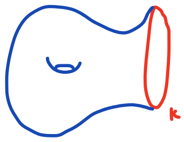


Which is the double of S:



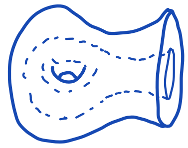
Hence $g(\partial \nu(S)) = 2g(S)$.

Picture to have in mind:

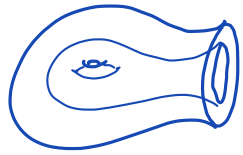


Seifert surface

→ thicken it up to a tubular nhood:



boundary of this looks like



Kind of like two copies of S stacked inside each other and connected by an annulus at the boundary

which has genus $2g(S)$ then.

Lemma: E_S is connected and $H_1(E_S) \cong \mathbb{Z}^{2g(S)}$

proof: write $S^3 = E_S \cup_{\partial V(S)} V(S)$. Mayer Vietoris sequence:

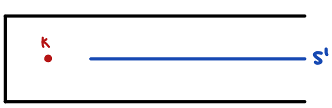
$$\underbrace{H_2(S^3)}_{=0} \to \underset{\mathbb{Z}^{4g(S)}}{\underset{\|}{H_1(\partial V(S))}} \xrightarrow{\sim} \overset{\mathbb{Z}^{2g(S)}}{H_1(V(S))} \oplus \underset{\mathbb{Z}^{2g(S)} \text{ (forced to be)}}{H_1(E_S)} \longrightarrow \underbrace{H_1(S^3)}_{=0}$$

$$\to \underset{\text{connected}}{\underset{\mathbb{Z}}{\underset{\|}{H_0(\partial V(S))}}} \longrightarrow \underset{\mathbb{Z}}{\underset{\|}{H_0(V(S))}} \oplus \underset{\mathbb{Z}}{\underset{\|}{H_0(E_S)}} \longrightarrow \underset{\mathbb{Z}}{\underset{\wr}{H_0(S^3)}}$$

⇒ E_S is connected ($H_0(E_S) \cong \mathbb{Z}$).

Lemma: $E_K \overset{\text{homeo}}{\cong} E_S/\sim$ where $i_+(x) \sim i_-(x)$ where $i_\pm : S \to S_\pm \subset \partial V(S)$ are the inclusions.

proof:



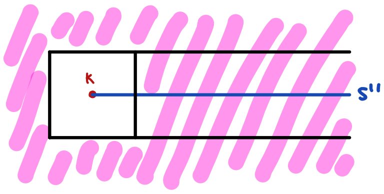
$$E_S/\sim \;\cong\; E_{S'}/\sim$$

$$\cong E_K/\sim' \quad , \text{ where } i(x,t) \sim i(x,t')$$

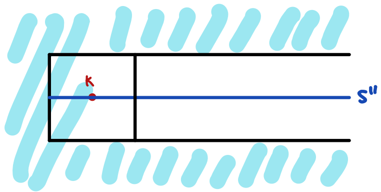
$i: S' \times [-1,1] \to S$ is inclusion.

Claim: $E_K \simeq E_S/\sim$, where $i_+(x) \sim i_-(x)$ $\subset \partial\nu(S)$ are the inclusions.

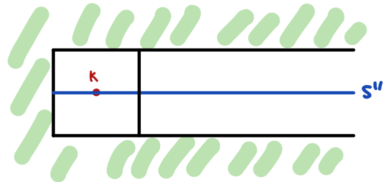
pf: Idea is that $E_S = E_K \setminus \partial\nu(S')$. But $E_S \simeq E_{S'} \simeq E_{S''}$. Let's look at what these look like:



E_K

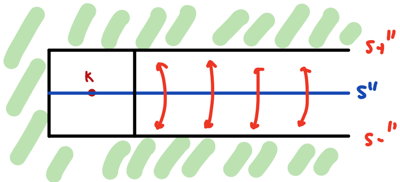


E_S



$E_{S''}$

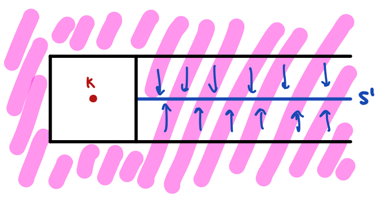
So I think the idea is you can stitch up $E_{S''}$ to make E_K:



identify S_+'' and S_-''

$E_{S''}$

Now we could equivalently do:



E_K

and squash the tubular neighbourhood of S'' onto S : i.e. $E_S/\sim \; \simeq E_K/\sim$, where our new relation is $i(x,t) \sim i(x,t')$, where $i: \nu(S') = S' \times [0,1] \to S^3$ is the inclusion

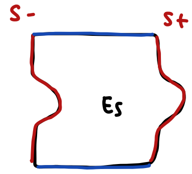
this is in fact homeomorphic to E_K. why is that? I need to ask Harley what he wrote, imagine its

26/02

Suppose $K \hookrightarrow S^3$, S is a Seifert surface and $E_S = S^3 \setminus V(S)$, $\partial E_S = S_+ \cup \partial S \times [-1,1] \cup S_-$

And denote $i_\pm : S \xrightarrow{\simeq} S_\pm$.

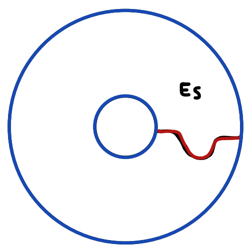
Schematic picture:



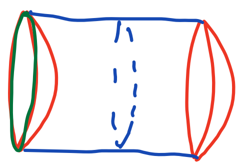
$\bullet = \partial S \times [-1,1]$

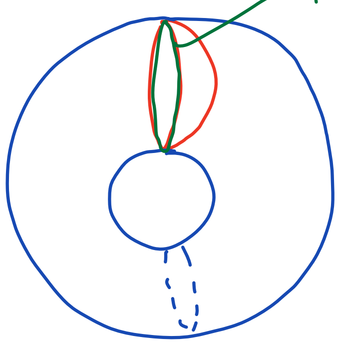
idea: E_S lives inside S^3,

Then $E_K \simeq E_S/\sim$ where $i_+(x) \sim i_-(x)$. Schematically:



Also showed $H_1(E_S) \simeq \mathbb{Z}^{2g}$, where $g = g(S)$.





forms a longitude for the space outside this donut.

And by thickening up arguments,

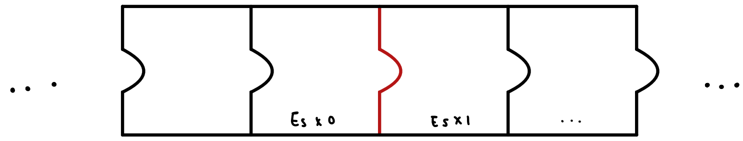
$\Rightarrow$ 0 lives inside whatever's outside this.

$\downarrow$

tubular nhood of 0

but 0 is exactly K.

Consider $Y = E_S \times \mathbb{Z}/\sim$, where $(i_+(x), n) \sim (i_-(x), n+1)$. This space looks like



boundary is really just $S^1 \times \mathbb{R}$. Then $\mathbb{Z}$ acts freely on Y by $k \cdot (x, n) = (x, n+k)$. Taking the quotient,

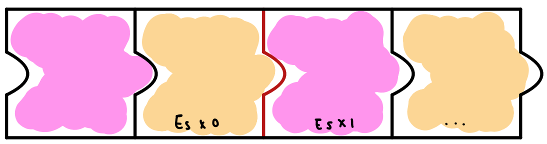
$$Y/\mathbb{Z} = E_S / i_+(x) \sim i_-(x) \simeq E_K$$

So projection $p: Y \to Y/\mathbb{Z} \simeq E_K$ is a covering map with Deck group $\mathbb{Z}$. Since $H_1(E_K)$ is $\mathbb{Z}$, there is only one such covering map with Deck group $\mathbb{Z}$, the infinite cyclic cover. That is, $Y \simeq \tilde{E}_K$.

Lemma: As a module over $R = \mathbb{Z}[H_1(E_K)] = \mathbb{Z}[t^{\pm 1}]$, $H_1(Y) \simeq \operatorname{coker}(t i_{-*} - i_{+*})$, where the maps $i_{\pm *}$ are the maps induced by $i_\pm$, $i_{\pm *}: H_*(S) \otimes R \longrightarrow H_*(E_S) \otimes R$

$\uparrow$ $i_\pm : S \to S_\pm$, and $S_\pm \subseteq \partial(\nu(S)) = \partial(E_S)$.

proof: Cut Y up into two bits: Going to use MV.



Let $E = \{n \in \mathbb{Z} : 2 \mid n\}$, $O = \{n \in \mathbb{Z} : 2 \nmid n\}$. Have projection map. $\pi: E_S \times \mathbb{Z} \to Y$

Let $A = \pi(E_S \times E)$ and $B = \pi(E_S \times O)$. Then $A \cup B = Y$, and $A \cap B \simeq S \times \mathbb{Z}$.

Mayer Vietoris:

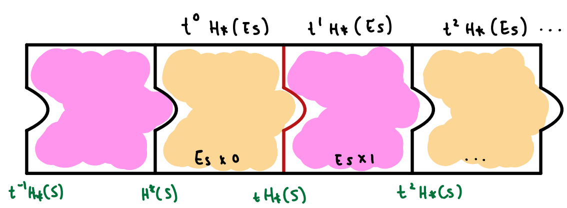
$$\longrightarrow H_*(S \times \mathbb{Z}) \xrightarrow{\overset{?}{i_{A*} - i_{B*}}} H_*(A) \oplus H_*(B) \longrightarrow H_*(Y)$$

We have that

- $H_*(S \times \mathbb{Z}) \simeq H_*(S) \otimes R$
- $H_*(A) \oplus H_*(B) \simeq H_*(E_S) \otimes R$

$S \times \mathbb{Z} = A \cap B$, which is really just the union of S and all the other copies of S induced by the action of G_{Deck}, $G_{Deck} = \langle t^i \rangle$. Hence, we have by Künneth (maybe?) that $H_*(S \times \mathbb{Z}) \simeq H_*(S) \otimes R$.

To see the second one, think about the following diagram:



I guess maybe from this then we can also really just see that $H_*(S \times \mathbb{Z}) \simeq H_*(S) \otimes R$. I think probably Künneth is needed to make sense of this though, since actually its a tensor over $\mathbb{Z}[t^{\pm 1}]$, not just $\mathbb{Z}$. Maybe?

Then MV becomes

$$\begin{tikzcd} H_1(S)\otimes R \arrow[r, "ti_{-*}-i_{+*}"] & H_1(E_S)\otimes R \arrow[r] & H_1(Y) \arrow[dll] \\ H_0(S)\otimes R \arrow[r, "ti_{-*}-i_{+*}"] & H_0(E_S)\otimes R & \end{tikzcd}$$

$\mathbb{Z}\otimes R = R \xrightarrow{t-1} R = \mathbb{Z}\otimes R$ (notice E_S and S connected $\Rightarrow H_0(S) = H_0(E_S) = \mathbb{Z}$.

The fact that he wrote $i_{A*} - i_{B*}$ is kind of stupid, but makes sense when you see that $H(S\times\mathbb{Z}) \simeq H_*(S)\otimes R$, and that $H_*(A)\oplus H_*(B) \simeq H_*(E_S)\otimes R$. The idea is that we can think about how they include.

$i_{A*} - i_{B*}$ is then $t\,i_{-*} - i_{+*}$

This map $R \to R$ is injective, and so $H_1(Y) \to H_0(S)\otimes R$ must be the zero map. So this says that $H_1(Y) = \operatorname{coker}(ti_{-*} - i_{+*})$.

Recall cokernel of $f: A \to B = {}^{B}/_{\operatorname{Im}(f)}$. Now the MV sequence becomes

$$H_1(S)\otimes R \xrightarrow{\alpha} H_1(E_S)\otimes R \overset{\beta}{\twoheadrightarrow} H_1(Y) \to 0 \qquad \alpha = ti_{-*} - i_{+*}$$

First iso says that $H_1(Y) \simeq {}^{H_1(E_S)\otimes R}/_{\ker\beta} \simeq {}^{H_1(E_S)\otimes R}/_{\operatorname{Im}(\alpha)} =: \operatorname{coker}(\alpha)$. $\square$

Theorem (Seifert): if K is a knot, then $\deg(\Delta_K t) \leq 2g(K)$, where $\deg \Delta_K(t)$ is the difference between the lowest and highest powers of t.

proof: if S is a seifert surface for K, then $H_1(\tilde{E}_K) \simeq \operatorname{coker}(ti_{-*} - i_{+*})$.

$\simeq \operatorname{coker}(tA_- - A_+)$, The maps i_{-*} and i_{+*} have no t in them.

where $A_\pm : \underset{\simeq \mathbb{Z}^{2g}}{H_1(S;\mathbb{Z})} \to \underset{\simeq \mathbb{Z}^{2g}}{H_1(E_S)}$ are $2g(S)\times 2g(S)$ matrices, with entries in $\mathbb{Z}$.

Denote: $B := tA_- - A_+$. Then B is a $2g\times 2g$ matrix whose entries are linear polynomials in t.
Hence $\Delta_K(t) \sim \det(B)$ is a poly. of degree $\leq 2g$.

We have $0 \to H_1(S)\otimes R \xrightarrow{B} H_1(E_S)\otimes R \to H_1(Y) \to 0$ is a presentation for $H_1(Y)$ as an R module, $R = \mathbb{Z}[H_1(E_K)] \simeq \mathbb{Z}\langle t^{\pm 1}\rangle$. And so $\Delta_K(t) \sim e_0(B) \sim \det(B)$
square, so no cols to delete

$\square$

The degree of $\Delta_K(t) \leq 2g(S)$. This is true for any S, so $\deg \Delta_K(t) \leq 2g(K)$.

Seiferts alg. always gives you minimal one actually.

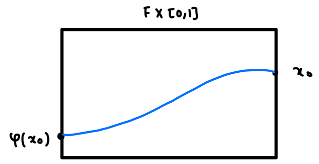
Remark: $2g(K) = \deg \Delta_K(t)$ if K is alternating or if K has ≤ 10 crossings.
: But not always true, e.g. knots on the gates of CMS have Alexander polynomial $\Delta_K(t) = 1 = \Delta(u)$, but $g(K_1) = 2$ and $g(K_2) = 3$.

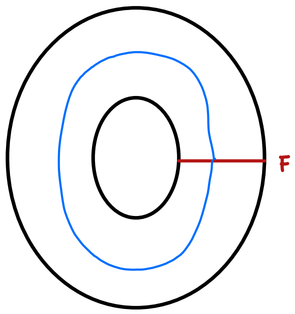
Fibred knots

Suppose E_K fibres over S^1 with connected fibre F

Lemma: F is a Seifert surface for K.

proof: $E_K \cong F \times [0,1]/\sim$, where $(\varphi(x), 0) \sim (x, 1)$ and $\varphi : F \xrightarrow{\cong} F$ (via the monodromy).
By hypothesis, $F \times [0,1]$ is connected. Fix $x_0 \in F$ and choose a path γ from $(\varphi(x_0), 0)$ to $(x_0, 1)$.





Then γ closes to give a loop in E_K:
and $[\gamma] \cdot F = 1 \Rightarrow F$ generates $H_2(E_K, \partial E_K)$
$\cong \mathbb{Z} \Rightarrow F$ is a Seifert surface.

Corollary: $g(K) = g(F)$

proof: $\Delta_K(t) \sim \det(\varphi_* - tI)$, where $\varphi_* : \underset{\mathbb{Z}^{2g(F)}}{H_1(F)} \to \underset{\mathbb{Z}^{2g(F)}}{H_1(F)}$ is an iso since φ is a diffeo. So
$\deg(\det(\varphi_* - tI))$ has degree $2g(F)$ (think about matrix), so $2g(F) \leq 2g(K)$, but F is
a Seifert surface. So actually $2g(F) = 2g(K)$.

$\square$

Corollary: if K is fibred, then $\Delta_K(t)$ is monic (highest power of t has coefficient 1).
This is iff if K is alternating or K has ≤ 10 crossings.

$\downarrow$
always true for a characteristic polynomial, like $\det(\varphi_* - tI)$

3. Knots and 3 & 4 manifolds

3.1 Handle bodies

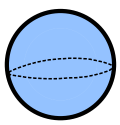
Definition: an n-dimensional k-handle is $\underset{\text{big}}{D^k} \times \underset{\text{small}}{D^{n-k}}$.

$D^k \times \{0\}$ is the core
$\{0\} \times D^{n-k}$ is the cocore
$S^{k-1} \times \{0\}$ is called the attaching sphere
$0 \times S^{n-k-1}$ is called the belt sphere

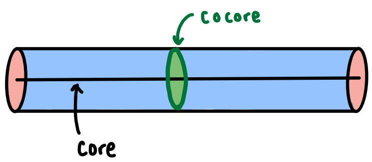
And $\partial H_n^k = \underbrace{\partial D^k \times D^{n-k}}_{\partial_A H_n^k} \cup_{S^{k-1} \times S^{n-k-1}} \underbrace{D^k \times \partial D^{n-k}}_{\partial_B H_n^k}$

Pictures for $n = 3$:

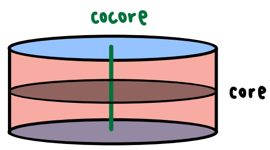
$k = 0$:



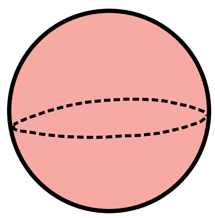
$k = 1$:



$k = 2$:

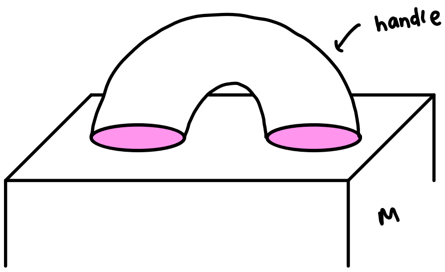


$k = 3$:



Basic fact: if M is a smooth n-manifold, with boundary, and $j : \partial_A H_n^k \hookrightarrow \partial M$ is an embedding, then $M \cup_j H_n^k =: M(j)$ is a smooth n-manifold with boundary.

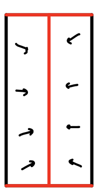
Picture:



$\mathrm{Im}(j)$ = attaching region

lemma: $\partial_A H_n^k \cup D^k \times \{0\}$ is a strong deformation retract of H_n^k

picture:



Lemma implies $M \cup_{j|_{S^{k-1}\times\{0\}}} D^k$ is a strong deformation retract of $M \cup_j H_n^k$ (they're homotopy equivalent spaces).

$\underbrace{S^{k-1}\times\{0\}}_{\text{attaching sphere}}$

Comparing Cell Complexes and handle bodies

Cell Complexes

$f: S^{k-1} \to X$

$X(f) = X \cup_f D^k$ add a k-cell

$f_0, f_1: S^{k-1} \to X$, $f_0 \sim f_1$

$\Rightarrow X(f_0) \simeq X(f_1)$

X is a finite n-dim. cell-complex rel $X_{-1} \subset X$ if there are subsets

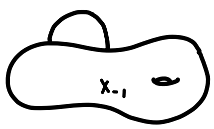
$$X_{-1} \subset X_0 \subset X_1 \subset \dots X_n = X$$

such that

$$X_k = X_{k-1}(F), \text{ where}$$

$F_i : \coprod_{i=1}^{n_k} S^{k-1} \to X$

I.e. $X_{k-1} \cup_F \coprod^{n_k} D^k$

E.g. 

attach a 1-cell.

Handle bodies

$j: \partial_A H_n^k \hookrightarrow \partial M$

$M(j) = M \cup_j H_n^k \simeq M \cup_{j|_{S^{k-1}\times\{0\}}} D^k$

Lemma: $j_0, j_1 : \partial_A H_n^k \hookrightarrow \partial M$ with $j_0 \sim_i j_1$, then $M(j_0) \overset{\text{diffeo}}{\cong} M(j_1)$.

pf: Isotopy $\Rightarrow$ ambient isotopy. So $\exists\ \varphi: M \to M$ a diffeo with $\varphi \circ j_0 = j_1$.

Then define $\Phi: M(j_0) \to M(j_1)$; $x \in M \longmapsto \varphi(x)$, $y \in H_n^k \longmapsto y$ clearly continuous, and actually a diffeo $\square$

Definition: an n-manifold M is a handlebody rel $M_{-1} \subset M$ (where M_{-1} is a closed m-dimensional submanifold with boundary) if there is a sequence

$$M_{-1} \subset M_0 \subset M_1 \subset \dots \subset M_n = M, \text{ where}$$

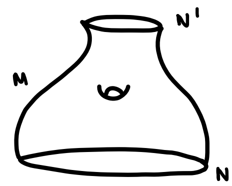
$M_k = M_{k-1}(J)$, where $J: \coprod^{n_k} \partial_A H_n^k \hookrightarrow \partial M$ (attach n_k k-handles at once)

M_k means the space with all $(\leq k)$-handles attached

By induction, easy to see that $M_k \simeq X_k$, where X_k is a cell complex rel M_{-1}, with k-handles $\leftrightarrow$ k-cells.

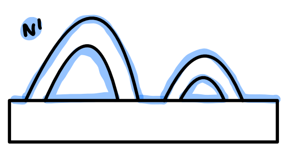
Slogan: (Morse - Smale) "all smooth manifolds are divided into handles".

Definition: If N, N' are (smooth) $(n-1)$-manifolds without boundary, a cobordism $M: N \to N'$ is a smooth n-manifold M with a diffeomorphism $g: \partial M \longrightarrow -N \sqcup N'$



e.g. $N \times [0,1] : N \to N$

Theorem (Morse/smale): If $M: N \to N'$ is a cobordism, then M is a handlebody rel $\nu(N) \cong N \times [0, \varepsilon]$ (N boundary so normal bundle trivial). Moreover, all handles are attached on $N \times \varepsilon$ boundary.



$N \times [0,\varepsilon]$

proved using Morse theory: choose a Morse function $f: M \to [0,1]$, with $f|_N \equiv 0$, $f|_{N'} \equiv 1$. Then

index k critical point $\longleftrightarrow$ k-handle

Chain Complex for Cells vs. Handles

Cellular Chain Complex:

If X is a cell complex rel X_{-1}, then $H_*(X, X_{-1}) \cong H_*^{cell}(X, X_{-1})$, where $C_k^{cell}(X, X_{-1})$ is generated by $e^1_1, \dots, e^k_{n_k}$, the k-cells of X rel X_{-1}. Then

$$de_i^k = \sum_j n_i^j e_j^{k-1}$$

Where n_i^j's are found by:

attaching map of e^k_i

$$\begin{tikzcd} S^{k-1} \arrow[r, "f_i"] & X_{k-1} \arrow[r] \arrow[dr, "f_i^j"] & X_{k-1}/X_{k-2} \simeq \bigvee^{n_{k-1}}_{\underbrace{}_{k-1 \text{ cells of } X}} S^{k-1} \arrow[d, "\pi_j"] \\ & & S^{k-1} \end{tikzcd}$$

resulting map

Then $n_i^j = \deg f_i^j$.

Cell Complex of a Handlebody

Suppose M is a handlebody rel M_{-1}. Then $M \sim X$ a cell complex rel M_{-1}, so

$$H_*(M, M_{-1}) \cong H_*(X, M_{-1}) \cong H_*^{cell}(X, M_{-1})$$

Question: what is $C_*^{cell}(X, M_{-1})$?

$C_k^{cell}(X, M_{-1})$ has generators $h^k_1, \dots, h^k_{n_k}$ corresponding to the k-handles ($H^k_{n,1} \dots H^k_{n,n_k}$) of M rel M_{-1}. But the boundary maps?

Let A^k_i = attaching sphere of $H^k_{n,i}$, i.e. $A^k_i = S^{k-1} \times \{0\} \subset \partial_A H^k_{n,i}$. Of course, $A^k_i \overset{homeo}{\simeq} S^{k-1}$.

Let B_j^{k-1} = belt sphere of $H^{k-1}_{n,j}$, i.e. $B^{k-1}_j = 0 \times S^{n-k} \subset \partial_B H_{n,j}^{k-1}$

$(n-(k-1))-1$

Rem: $A^k_i, B_j^{k-1} \subset \partial M_{k-1}$. M_k = space with all $(\leq k-1)$-handles attached

So $A_i^k \simeq S^{k-1} \subset \partial M_{k-1}$ (∂M_{k-1} is an $(n-1)$-manifold)

$B_j^{k-1} \simeq S^{n-k} \subset \partial M_{k-1}$

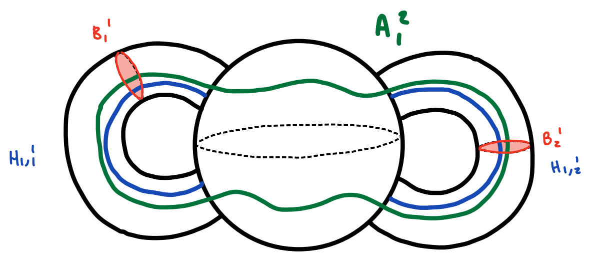
transverse intersection $\to \dim(A_i^k \cap B_i^{k-1}) = 0$

By dimension reasons $((k-1)+(n-k) = n-1)$, they generically intersect in points and have a well defined intersection #.

Lemma: $dh_i^k = \sum n_i^j h_j^{k-1}$, where $n_i^j = A^k_i \cdot B^{k-1}_j$ (intersection number in ∂M_{k-1})

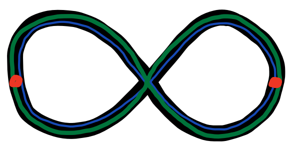
Sketch of proof (no signs): A^k_i is the image of the attaching map $f_i^k : S^{k-1} \hookrightarrow \partial M_{k-1}$

The picture that we get is (e.g. for $n=3$, $k=2$)



Say A_1^2 is the attaching sphere for the 2-handle $H^2_{3,1}$, $A_1^2 \simeq S^1$ and embeds into ∂M_1, M_1 is the union of the original space with all 0 and 1 handles. Our space M_1 in the picture is a 0-handle (the central ball), with two 1-handles $H_{1,1}'$ and $H_{1,2}'$. Their cores are given in blue, and their belt spheres in red.

All of this stuff deformation retracts onto the figure 8: collapsing 0-handle to point and 1-handles to their cores:

$\sim$ 

(drawn it a bit chubby but really 1-dim)

Really, the idea is to follow the exact same process as for the cellular case. So we're interested in the degree of the attaching map of A_1^2 at this stage. How do we compute this? Well you look at a generic point $p \in \vee S^1$, and count # of points in the preimage of p under the attaching map f_i. I.e. $n_i^1 = \#\text{pts in } (f_i^2)^{-1}(p) = \underset{\text{cocore of } H_1^1}{\underset{\shortparallel}{B_1^1}} \cap A_1^2$,

Have a map

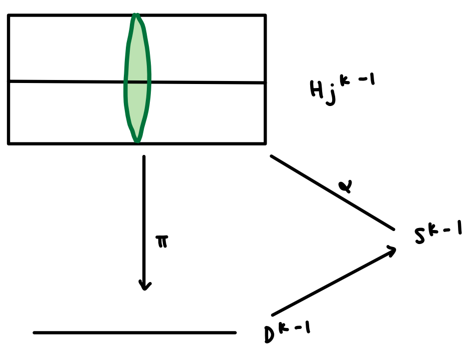
$$\begin{tikzcd}
S^{k-1} \arrow[r, "f_i"] \arrow[ddrrrr, "f_i^j"'] & \partial M_{k-1} \arrow[r, hook] & M_{k-1} \arrow[r, "\text{hom. equiv.}"] \arrow[d] & X_{k-1} \arrow[r] & X_{k-1}/X_{k-2} \arrow[dd, "\pi_j"] \\
& & M_{k-1}/(M_{k-1} \setminus H_{n,j}^{k-1}) \arrow[drr] & & \\
& & & & S^{k-1}
\end{tikzcd}$$

So $n_i^j = \deg(f_i^j)$

can factor these maps as in middle.

Hence $n_i^j = \#(f_i^j)^{-1}(p)$ if f is transverse at p.

Picture:



And so $(f_i^j)^{-1}(p) = f_i^{-1}(\pi^{-1}(p))$ $\qquad \pi^{-1}(p) = \text{cocore at } p=0$

$(\text{at } p=0) \quad = A_i^k \cap B_j^{k-1}$

f_i^j is transverse $\Leftrightarrow$ A_i^k intersects B_j^{k-1} transversally. $\square$

Cobordisms:

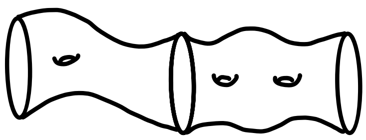
$M : N \to N'$ means $\exists$ diffeo $g : \partial M \to \bar{N} \sqcup N'$, where $\bar{N}$ means reversed orientation on N. Then $\partial \bar{M} = N \sqcup \bar{N}'$, so $\bar{M} : N' \to N$ (orientation reversal).

If $M_0 : N \to N'$ and $M_1 : N' \to N''$, then I have $M_1 \circ M_0 : N \to N''$ where $M_1 \circ M_0 = M_0 \cup_{N'} M_1$

Picture

$M_0 : N \to N'$ $\qquad$ $M_1 : N' \to N''$

Then $M_1 \circ M_0$ =

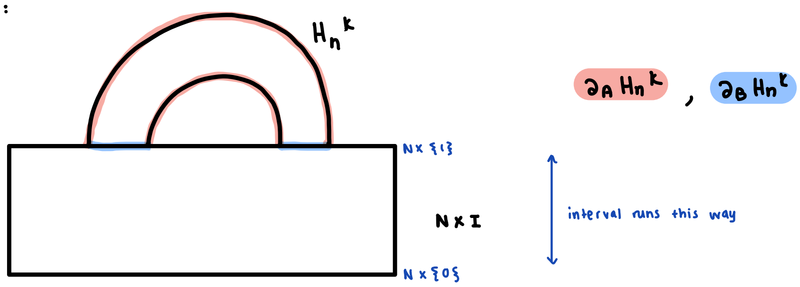


Exercise: $(M_2 \circ M_1) \circ (M_0) \cong M_2 \circ (M_1 \circ M_0)$

Surgery

Definition: Suppose N is an $(n-1)$-manifold, and $j : \partial_A H_n^k \hookrightarrow N$. Let $N[j] = N \times I (j \times 1) = N \times I \cup_{j \times 1} H_n^k$

Picture: $N[j]$ looks like:



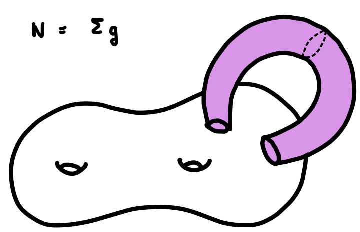
Basically $N \times I$ with handle H_n^k attached on $N \times \{1\}$.

Then $\partial[j] = \bar{N} \times \{0\} \sqcup (N \setminus \mathrm{int}(im j)) \cup_{j(S^{k-1} \times S^{n-k-1})} \partial_B H_n^k$

Hence $N[j]$ is a cobordism from N to N', where $N' = (N \setminus \mathrm{int}(im j)) \cup_{j(S^{k-1} \times S^{n-k-1})} \partial_B H_n^k$

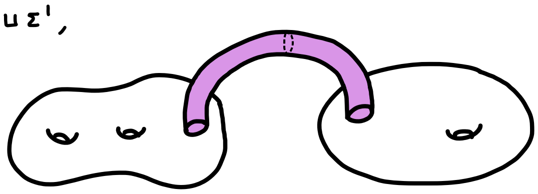
We say N' is the result of surgery on N along j. The cobordism $N[j]$ is called the trace of the surgery

Ex: $n=3$: 1) add a 1-handle, $N = \Sigma_g$



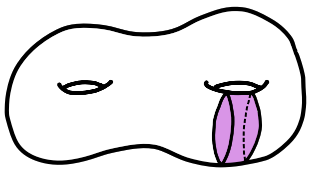
Idea: thicken up $N = \Sigma_g$, and to outside add a 3-dim 2-handle. Then outside boundary is Σ_{g+1}, and inside boundary is still Σ_g. So $N[j] : \Sigma_g \to \Sigma_{g+1}$

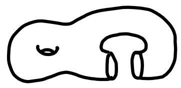
2) add a 1-handle, $N = \Sigma \sqcup \Sigma'$,



Again using the image of thickening up the space N. Inside boundary is still $\Sigma \sqcup \Sigma'$, but outside boundary is now $\Sigma \# \Sigma'$. So we get a cobordism $N[j] : \Sigma \sqcup \Sigma' \to \Sigma \# \Sigma'$.

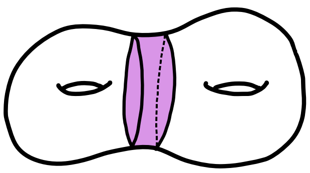
3) add a 2 handle, $N = \Sigma_g$



Outer boundary: stays the same. Inner boundary looks like  $\simeq$ , i.e Σ_{g-1}

Get cobordism $N[j] : \Sigma_g \to \Sigma_{g-1}$

Could also attach it like:



Same shpiel: $N[j] : \Sigma \# \Sigma' \to \Sigma \sqcup \Sigma'$.

If $N' = (N \setminus \mathrm{int}(\mathrm{im}(j))) \cup_{j(S^{k-1} \times S^{n-k-1})} \partial_B H_n^k$, then $j' : \underset{\underset{S^{n-k-1}}{\cong}}{\partial_B H_n^k} \hookrightarrow N'$ is an embedding. But

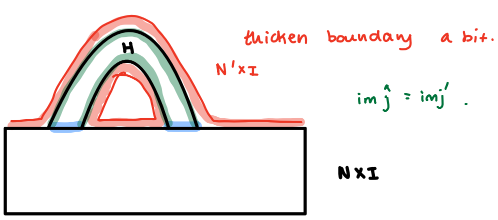
$H_n^k = D^k \times D^{n-k} \overset{\text{diffeo}}{\simeq} D^{n-k} \times D^k =: H_n^{*\,n-k}$, and $\partial_A H_n^k = \partial_B H_n^{*\,n-k}$, and $\partial_B H_n^k = \partial_A H_n^{*\,n-k}$.

So

$$\begin{tikzcd} \partial_B H_n^k \arrow[r, hook, "j'"] & N' \\ \partial_A H_n^{*\,n-k} \arrow[u, phantom, "\cong" rotate=90] \arrow[ur, "\hat{j}"'] & \end{tikzcd}$$

lemma: $N'[\hat{j}] \overset{\text{diffeo}}{\simeq} \bar{N}[j]$., $N'[\hat{j}] : N' \to N$, and $N[j] : N \to N'$ so $\bar{N}[j] : N' \to N$.

Picture:



Proof: $\overline{N[j]} \overset{\text{diff}}{\simeq} \overline{N[j]} \cup_{N'} N' \times I$

$$\overset{\text{diff}}{\simeq} \overline{N \times I} \cup_j H_n^k \,{}_j\!\cup_{N'} N' \times I$$

$$\overset{\text{diff}}{\simeq} N \times I \cup \left(H_n^{*\,n-k} \cup_{\hat{j}} \overline{N' \times I} \right)$$

$$\overset{\text{diff}}{\simeq} N \times I \cup_N N'[\hat{j}]$$

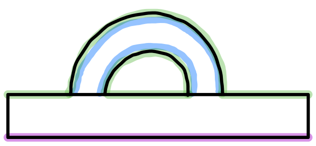
need to fix the orientations.

I think blue fixes it

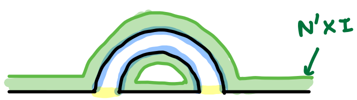
really this is kind of just handle cancellation

Let's draw a picture so we can solidify what surgery really is. Attaching region: $S^{k-1} \times D^{n-k}$, and belt region is $D^k \times S^{n-k-1}$. i.e. these are $\partial_A H_n^k$ and $\partial_B H_n^k$ respectively so the cobordism has boundary components $\bar{N}$ and N', where N' is the result of doing surgery on N. We cut out the $(n-1)$-dimensional manifold, $\partial_A H_n^k \simeq S^{k-1} \times D^{n-k}$, taking the closure and then gluing back the $(n-1)$-dim. manifold $D^k \times S^{n-k-1}$

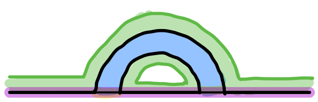
Okay so the idea is as follows. We have our cobordism $N[j] : N \to N'$;



Bottom purple boundary is N, and top green boundary is N'. Now actually, in blue we have the (image of) the boundary $\partial_B H_n^k$. Now, we think of $H_n^k \simeq H_n^{n-k*}$, so that $\partial_B H_n^k$ can be thought of as the attaching region for H_n^{n-k}. Now we think about what $(n-k)$-surgery does: we first thicken up N':

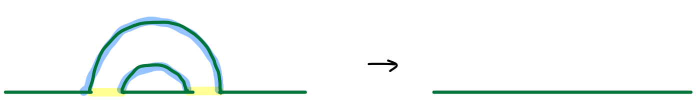


Then we attach in the H_n^{n-k*} handle, and the resulting manifold is



We see there's 2 boundary components. The green, which is our original N', and the purple, which is actually N. Let's make this more formal by looking at what surgery does to the boundary:

cut out $\partial_A(H_n^{n-k*})$, $\simeq S^{n-k-1} \times D^{n-k}$, i.e. the blue part, which leaves those gaping wounds, and we glue back $\partial_B(H_n^{n-k*})$, which looks like $D^{n-k} \times S^{n-k-1}$, which is the yellow parts. The result:



Corollary: Suppose $M: N \to N'$ is a handlebody, so $M = M(r) \circ M(r-1) \circ \cdots \circ M(1)$ is a composition of traces of surgeries, with k-handles $H_{n,1}^{k}, \ldots, H_{n,n_k}^{k}$, $r = \sum_{i=0}^{n} n_k$
Then $\bar{M} : N' \to N$ is a handle body with $(n-k)$-handles $H_{n,i}^{*n-k}$, dual to $H_{n,i}^{k}$.

Theorem: if $M: N \to N'$ is a handle body of dimension n, then

$$H_k(M, N) \simeq H^{n-k}(M, N') \quad \text{and} \quad H^k(M,N) \simeq H_{n-k}(M, N')$$

This is known as Poincaré-Lefschetz duality.

Proof: (will do with $\mathbb{Z}/2$ coeffs, but works with $\mathbb{Z}$ coeffs if M is orientable)

Consider $H_*(M, N; \mathbb{Z}_2) \simeq H_*^{cell}(M, N; \mathbb{Z}_2)$, where

$C_k^{cell}(M, N; \mathbb{Z}_2) = \langle h_1^k, \ldots, h_{n_k}^k \rangle$ (generated by k-handles), and $d h_i^k = \sum_j n_{ij} h_j^{k-1}$, where

$n_{ij} = A^k_i \cdot B_j^{k-1}$ (intersection number) in ∂M_{k-1}.

On the other hand, considering the dual handle decomposition, $\bar{M} : N' \to N$, we see that

$$H_*(M, N'; \mathbb{Z}_2) \simeq H_*^{cell}(M, N'; \mathbb{Z}_2),$$

where $C_{n-k}^{cell}(M, N')$ is generated by $\langle h_1^{*n-k}, \ldots, h_{n_k}^{*n-k} \rangle$, with $dh_j^{*n-k} = \sum n_{ij}' h_i^{*n-k+1}$, where

a_k

$n_{ij}' = A_j^{*n-k} \cdot B_i^{*n-k+1}$

$h_i^{*n-k} \simeq h_i^k$

But $A(H_{n,i}^k) = B(H_{n,i}^{*n-k})$, and $B(H_{n,i}^k) = A(H_{n,i}^{*n-k})$. So $n_{ij}' = n_{ij}$, i.e.
$C_*^{cell}(M, N; \mathbb{Z}_2)$ is dual to $C_*^{cell}(M, N'; \mathbb{Z}_2)$, hence isomorphic to $C^*_{cell}(M, N'; \mathbb{Z}_2)$. $\square$
$C_k^{cell}(M, N; \mathbb{Z}_2)$ is dual to $C_{n-k}^{cell}(M, N'; \mathbb{Z}_2)$

Remark: We have proved "weak Poincaré duality":

$$H_k(M, N) \simeq H^{n-k}(M, N')$$ with $\mathbb{Z}_2$ coeffs or with $\mathbb{Z}$ coeffs if M is orientable.

Strong Poincaré duality: for a field $\mathbb{F}$,

$$\cup : H^k(M, N; \mathbb{F}) \times H^{n-k}(M, N'; \mathbb{F}) \longrightarrow H^n(M, N \cup N'; \mathbb{F})$$

$$\cong H^n(M, \partial M; \mathbb{F}) \xrightarrow{\langle \cdot, [M] \rangle} \mathbb{F}$$

Is a nonsingular pairing for any field $\mathbb{F}$ if M is orientable.

3.2. The Seifert Matrix

Recall: if $M : N \to N'$ is a orientable, n-dim cobordism, then P.D. iso exists mapping

$$P.D: H_k(M,N) \xrightarrow{\sim} H^{n-k}(M,N')$$

Useful special case: $M : \emptyset \to \partial M$, so any manifold is a cobordism from the empty $(n-1)$-manifold to its boundary.

$$P.D: H_k(M,\partial M) \xrightarrow{\sim} H^{n-k}(M)$$
$$P.D: H_k(M) \xrightarrow{\sim} H^{n-k}(M,\partial M)$$

Suppose $K \hookrightarrow S^3$ is a knot, and $S \hookrightarrow S^3$ is a Seifert surface of K. Then $\Delta_K(t) \sim \det(A^+ - tA^{-1})$, where $A^{\pm}$ are matrices representing the maps $i_{\pm}^*: H_1(S) \to H_1(E_S) \simeq \mathbb{Z}^{2g(S)}$

Lemma 1: $H_1(E_S) \simeq H^1(S)$.

Proof: $H_1(E_S) \underset{a)}{\simeq} H^2(E_S, \partial E_S) \underset{b)}{\simeq} H^2(S^3, \nu(S)) \underset{c)}{\simeq} H^1(\nu(S)) \underset{d)}{\simeq} H^1(S)$

a) is D.D

b) is excision (remove interior of $\nu(S)$)

going $\leftarrow$: $H^2(S^3, \nu(S)) \simeq H^2(S^3 \setminus \mathrm{Int}(\nu(S)), \nu(S) \setminus \mathrm{Int}(\nu(S))) \simeq H^2(E_S, \partial\nu(S))$

c) follows from LES of $(S^3, \nu(S))$, since $H^1(S^3) = H^2(S^3) = 0$.

LES looks like

$$\to H^1(X) \xrightarrow{i^*} H^1(A) \xrightarrow{\partial} H^2(X,A) \xrightarrow{q^*} H^2(X) \xrightarrow{i^*} H^2(A) \to \cdots$$

For $(S^3, \nu(S))$: $\underset{0}{\underset{\|}{H^1(S^3)}} \to \underset{\mathbb{Z}^{2g}}{\underset{\|}{H^1(\nu(S))}} \underset{\text{by exactness}}{\xrightarrow{\sim}} H^2(S^3, \nu(S)) \to \underset{0}{\underset{\|}{H^2(S^3)}}$

So $H^2(S^3, \nu(S)) \simeq \mathbb{Z}^{2g}$.

d) $S \sim \nu(S)$ (homotopy equivalent): tubular nhood is trivial. Remember orientable + rank 1 (line) bundle. $\square$

Consider $\alpha : H_1(E_S) \to H^1(S)$ as in the lemma:

Let $\alpha = \delta^{-1} \circ P.D$, where $P.D: H_1(E_S) \to H^2(E_S, \partial E_S)$, and $\delta^{-1} = d \circ c \circ b$, i.e. $\delta : H^1(S) \xrightarrow{\sim} H^2(E_S, \partial E_S)$ is the composition

$$H^1(S) \xrightarrow{\pi^*} H^1(\nu(S)) \longrightarrow H^2(S^3, \nu(S)) \longrightarrow H^2(E_S, \partial E_S)$$

Every group is free over $\mathbb{Z}$, so δ is dual to $\partial : H_2(E_S, \partial E_S) \longrightarrow H_1(S)$ given by the composition

$$H_2(E_S, \partial E_S) \underset{\text{excision}}{\longrightarrow} H_2(S, \nu(S)) \underset{\text{boundary in LES}}{\longrightarrow} H_1(\nu(S)) \xrightarrow[\pi_*]{} H_1(S)$$

If $\underset{\text{a surface, representing a homology class.}}{\Sigma} \hookrightarrow S^3$, $\partial\Sigma \subset S$, then $\partial[\Sigma, \partial\Sigma] = [\partial\Sigma] \in H_1(S)$ by chasing through maps.

Lemma 2: If $x \in H_1(E_S)$ and $y \in H_1(S)$ are represented by embedded circles (can always arrange this for classes in H_1), then $\langle \alpha(x), y\rangle = \ell k(x,y)$ in S^3.

proof: $\langle \alpha(x), y\rangle = \langle \delta^{-1} \circ PD(x), y\rangle = \langle PD(x), \partial^{-1}(y)\rangle$

dual map from above

Choose $\Sigma \hookrightarrow S^3$, with $\partial\Sigma = y$. Then $\partial[\Sigma, \partial\Sigma] = [\partial\Sigma] = y$, so

boundary of surface

$= \langle P.D(x), [\Sigma, \partial\Sigma]\rangle$

$= x \cdot \Sigma$ intersection pairing dual to cup pairing eval.

$= \ell k(x,y)$ from first example sheet.

since $\partial\Sigma = y$.

$\square$

via Poincaré duality

Bases: let $\{x_1, \dots, x_{2g}\}$ a basis of loops for $H_1(S)$. Then $\{x^1, \dots, x^{2g}\}$ the dual basis of $H^1(S)$, defined by $\langle x^i, x_j\rangle = \delta^i_j$. Then $\{y_1, \dots, y_{2g}\}$, $y_i = \alpha^{-1}(x^i)$ is a basis of $H_1(E_S)$.

Lemma 3: If $z \in H_1(E_S)$, then $z = \sum_{i=1}^{2g} \ell k(z, x_i)\, y_i$.

proof: $\alpha(z) \in H^1(S)$, so $\alpha(z) = \sum \langle \alpha(z), x_i\rangle x^i$ literally just from how we decompose w.r.t. basis bc. $\langle x^i, x_j\rangle = \delta^i_j$

$= \sum \ell k(z, x_i)\, x^i$ by lemma 2

$\alpha: H_1(E_S) \to H^1(S)$

$\Rightarrow z = \alpha^{-1}\left(\sum \ell k(z, x_i)\, x^i\right)$

$= \sum \ell k(z, x_i)\, \alpha^{-1}(x^i)$

$= \sum \ell k(z, x_i)\, y^i$.

$\square$

Let $A^{\pm} = [a_{ij}^{\pm}]$ be the matrix of the map $i_{\pm *}: H_1(S) \to H_1(E_S)$ w.r.t. the bases $\{x_1, \dots, x_{2g}\}$ and $\{y_1, \dots, y_{2g}\}$.

So $i_{\pm *}(x_j) = \sum a_{ij}^{\pm} y_i$ → y_i form basis for $H_1(E_S)$, so we can surely write whatever x_i gets mapped to as a linear combination of the y_i's.

Corollary: $a_{ij}^{\pm} = \ell k(i_{\pm *}(x_j), x_i)$

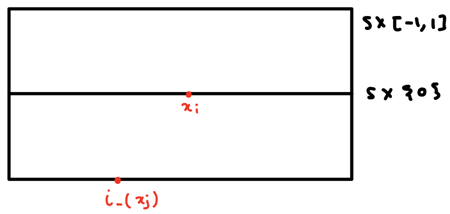
proof: follows by putting $z = i_{\pm *}(x_j)$ in lemma 3 and equating coefficients.

$\square$

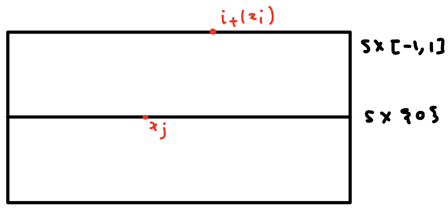
Corollary: $a_{ij}^- = a_{ji}^+$, i.e. $A^- = (A^+)^T$

proof: $a_{ij}^- = lk(i_-(x_j), x_i)$, and $a_{ij}^+ = lk(i_+(x_j), x_i)$

Schematic picture of $V(S) = S \times [-1,1]$



The link $i_-(x_j) \cup x_i \subset V(S) \subset S^3$ is isotopic to the link $x_j \cup i_+(x_i)$



just by shifting everything up. So $a_{ij}^- = lk(i_-(x_j), x_i) = lk(i_+(x_i), x_j) = a_{ji}^+$ $\square$

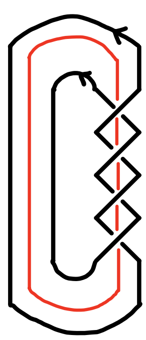
Definition: $A = A^+$ is the Seifert matrix of K determined by S and $\{x_1, \dots, x_{2g}\}$. Then

$$\Delta_K(t) \sim \det(A^+ - tA^-) = \det(A - tA^T).$$

Examples: How to compute Seifert matrix.

Key example is the k-twisted band $\qquad$ $K = 2$, $\qquad$ S = annulus

x = red curve



Then $H_1(S) = \langle x \rangle$, $\quad x = S \times \{0\}$.

oriented parallel to each other

Then $\quad lk(i_+(x), x) = k$, $\quad$ where $\quad k = lk(\partial_1 S, \partial_2 S) = \frac{1}{2} w(D)$

proof: $v(x) \simeq S^1 \times D^2$, and $\partial_1(S), \partial_2(S)$ are two sections of $V_{S^3/x}\big|$ unit sphere bundle. That is, $S = s \subset V$, where s is the total space of a section of the normal bundle.

> key thing to think about here is that $v(x) \simeq S^1 \times D^2$, a solid torus. We can think about $\partial_1(S)$ and $\partial_2(S)$ as sections of $S(v(x))$, which are closed simple curves that lie on the boundary $S^1 \times S^1$. Hence our surface lies in the tubular neighbourhood $v(x)$, as the total space of a section of the normal bundle.

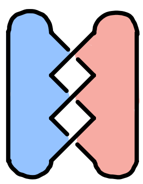
Hence $i_+(x)$ and $i_-(x)$ are sections of $V_{S/x}$ which are perpendicular to S (push above and below S)

$\Rightarrow$ $i_\pm(x)$ are homotopic to $\partial_1(S)$ and $\partial_2(S)$ respectively. So

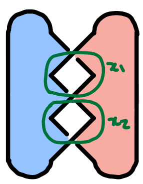
$$\begin{aligned} \ell k(i_\pm(x), x) &= \ell k(\partial_1(S), x) \\ &= \ell k(\partial_1(S), \partial_2(S)) = k. \end{aligned}$$

$\square$

Example: K = +ve trefoil



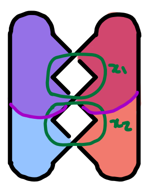
Choose basis for $H_1(S)$. Now $S \simeq$ punctured torus $T^2 \setminus \{pt\}$, and $H_1(T^2 \setminus \{pt\}) \simeq \mathbb{Z}^2$, so we need to find two basis loops. Most obvious choice:



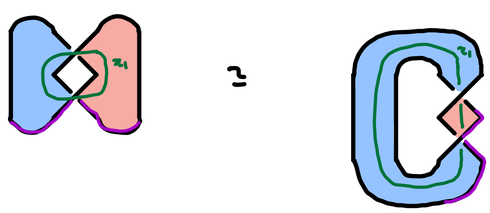
really x_1 and x_2 intersect at midpoint of middle band.

Wrt. $\{x_1, x_2\}$, the Seifert matrix is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Hence $\ell k(\iota_+(x_1), x_1) = 1$, see since $v(x_1) \subset S$, in purple below, is exactly a twisted band with $k = 1$. (two positive crossings). I.e. $v(x_1) \subset S$ is a twisted band.

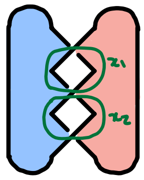


Here's what I think is going on. We look at the $\ell k(\iota_+(x_1), x_1)$ by considering it locally (don't have to worry about whole surface, just a sufficiently large nhood. Now we can take it to be the purple region, which is actually a twisted annulus ($k = 1$):



And so the idea is you can homotope x_1 onto one of the boundary components, lift up x to get $i_+(x)$, and then this guy is homotopic in S^3 to the other boundary component, so $\ell k(i_+(x), x) = \ell k(\partial_1(S), \partial_2(S)) = k = 1$ in this instance.

Similarly $\ell k(i_+(x_2), x_2) = 1$. To compute $\ell k(i_\pm(x_1), x_2)$. Now, x_1 and x_2 intersect in one point. When you push x_1 off in one direction, it's going to wind up linking once, and in the other direction it will not link at all.



Remember $a_{ij}^+ = a_{ji}^-$, so we can find the off diagonal entries by thinking about pushing say x_1 off S in both directions, and then seeing the resulting curves' linking number with x_2

Check: $\det(A - tA^T) = \det\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - t\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 1-t & 1 \\ -t & 1-t \end{pmatrix}$

$$= 1 - t^2 + t$$
$$\sim t^2 - t + 1 \sim \Delta_K(t)$$

Symmetry: Write $t = q^2$, so $\det(A - q^2 A^T) \underset{\text{up to mult power of } q}{\sim} \det(q^{-1}A - qA^T)$

Let $\hat{\Delta}_K(q) = \det(q^{-1}A - qA^T) = \det((q^{-1}A - qA^T)^T) = \det(-qA + q^{-1}A^T) = \Delta_K(-q^{-1})$.

So $\hat{\Delta}_K(q) = \hat{\Delta}_K(-q^{-1})$. So $\Delta_K(t)$ can be normalized, and so is symmetric under $t \mapsto t^{-1}$.

$t = q^2$, and $q \to -q^{-1} \Rightarrow q^2 \to (-q^{-1})^2 = q^{-2}$
$\Leftrightarrow t \leftrightarrow t^{-1}$

Normalized Alexander Polynomial

$L \hookrightarrow S^3$ an oriented link, and $S \hookrightarrow S^3$ a Seifert surface for L, with $\langle x_1, \dots, x_k \rangle = H_1(S)$. Then this data determines a Seifert matrix A

Definition: $\hat{\Delta}_L(q) = \det(q^{-1}A - qA^T)$

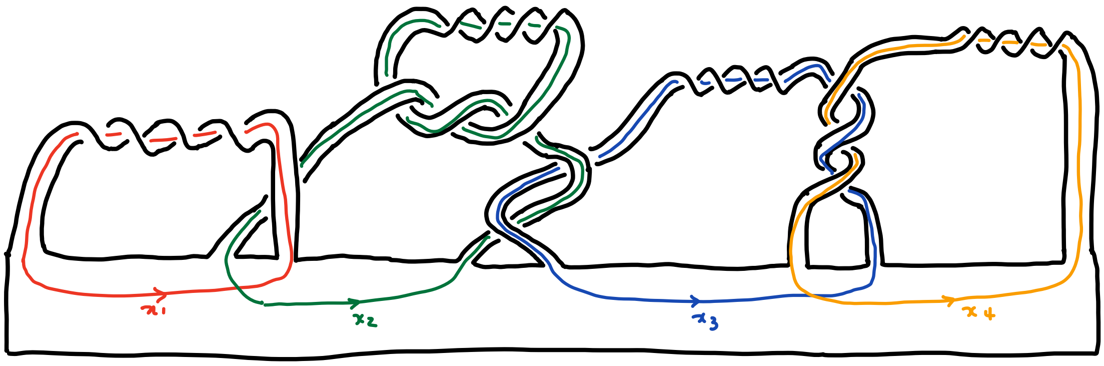
Last lecture:

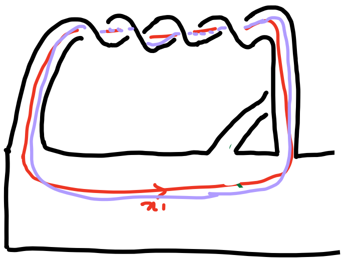
- $\hat{\Delta}_K(q) \sim \Delta_K(q^2)$
- $\hat{\Delta}_L(-q^{-1}) = \hat{\Delta}_L(q)$ (symmetry)

So symmetry determines $\hat{\Delta}(q)$ up to a sign.

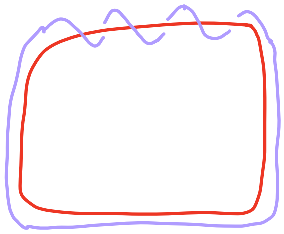
e.g $\Delta_{T(2,3)}(t) \sim t^2 - t + 1 \Rightarrow \hat{\Delta}_{T(2,3)}(q) \sim \pm(q^2 - 1 + q^{-2})$

Take the following Seifert surface:





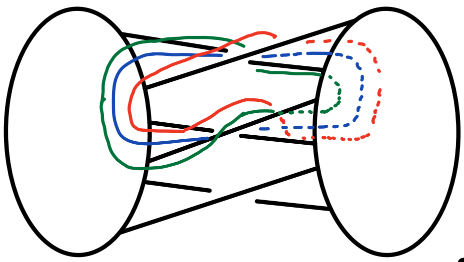
picture :



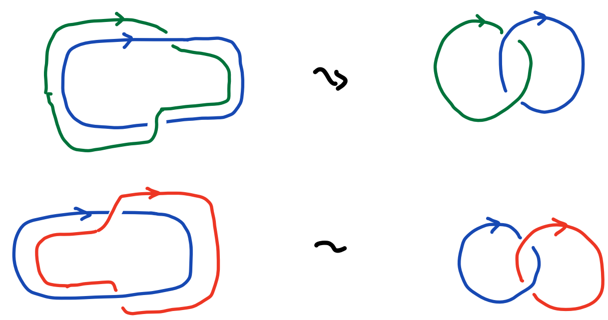
→ linking # = 2

linking # = $\frac{1}{2}$ (+ve crossings - (-ve) crossings).

green and red are both blue pushed up.



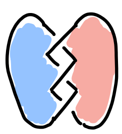
⇝



equiv.

Examples :

+ve Trefoil



⇝ Seifert matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

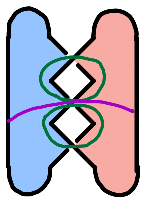
$$\hat{\Delta}(q) = \det\left(\begin{bmatrix} q^{-1} & q^{-1} \\ 0 & q^{-1} \end{bmatrix} - \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$= \det \begin{pmatrix} -(q-q^{-1}) & q^{-1} \\ -q & -(q-q^{-1}) \end{pmatrix} = (q-q^{-1})^2 - 1$$

$$= q^2 - 1 + q^{-2}$$

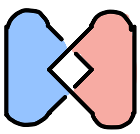
Unknot:

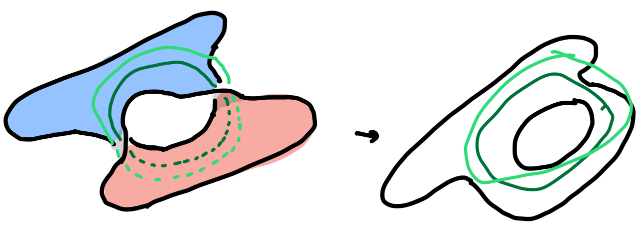
Seifert surface for unknot.



has seifert matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

looks like



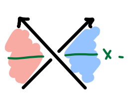


All the entries of A are the same as the previous example, except the 2,2 entry. If we consider the purple neighbourhood of the second loop, it's a band with two twists, which is just the same as a band with zero twists, and here the loop is trivial. So it has self linking number 0.

Check: $\hat{\Delta}(q) = 1$.

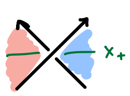
Proposition: $\underset{D_-}{\hat{\Delta}(\times)} - \underset{D_+}{\hat{\Delta}(\times)} = (q - q^{-1})\, \underset{D_0}{\hat{\Delta}(\uparrow\uparrow)}$ (Conway Skein relation)

proof: Apply Seifert's algorithm to get Seifert surfaces $S_\pm$, S_0 for $D_\pm$, D_0. Now $S_\pm$ are obtained by adding a 1-handle with a $\pm$ twist to S_0

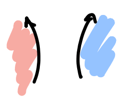
S_- 

If $H_1(S_0) = \langle x_1, \dots, x_k \rangle$

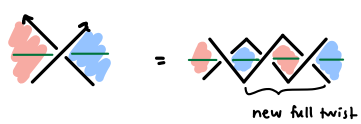
$H_1(S_\pm) = \langle x_1, \dots, x_k, x_\pm \rangle$

S_+ 

Claim: $\ell k(L_+(x_+), x_+) = \ell k(L_+(x_-), x_-) + 1$

S_0 

pf: If V_{S_-/x_-} is a k-twisted band, then



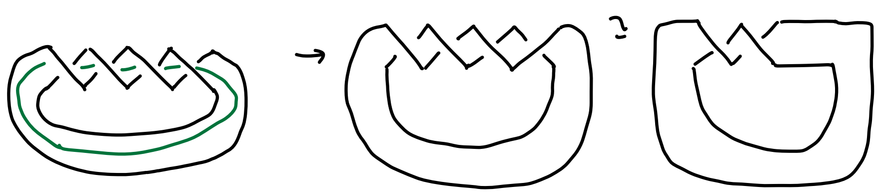
i.e. V_{S_+/x_+} is a band with $k+1$ twists

$\square$

$\Rightarrow$ We have Seifert matrices

$$A_\pm = \begin{bmatrix} A_0 & x \\ y & k \pm 1/2 \end{bmatrix}$$

Picture:

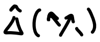
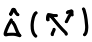
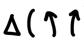


Then $\hat{\Delta}_{D_\pm}(q) = \det(q^{-1}A_\pm - qA_\pm^T)$

$$= \det \overbrace{\begin{pmatrix} q^{-1}A_0 - qA_0^T & z \\ w & (q^{-1}-q)(K \pm 1/2) \end{pmatrix}}^{B_\pm}$$

Expand det. along bottom row: all terms are the same except the last one.

$\Rightarrow \det(B_+) - \det(B_-) = (q^{-1}-q)\det(q^{-1}A_0 - qA_0^T)$

$\Rightarrow$ $\hat{\Delta}$() $-$ $\hat{\Delta}$() $= (q^{-1}-q)\,\Delta$() $\square$

Corollary: $\hat{\Delta}_K(1) = \hat{\Delta}_K(u) = 1$ (Exercise on example sheet)

$\Rightarrow$ $\hat{\Delta}_K$ is fully determined by $\Delta_K(t)$, i.e. it does not depend on the choice of Seifert surface S or the basis $\{x\}$.

Examples:

1) $\hat{\Delta}$() $-$ $\hat{\Delta}$() $= (q-q^{-1})\hat{\Delta}$()

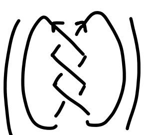
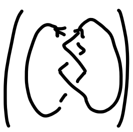
$= 1$ $\qquad$ $= 1$

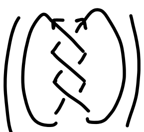
so $\hat{\Delta}$() $= 0$

2) $\hat{\Delta}$() $-$ $\hat{\Delta}$() $= (q-q^{-1})\hat{\Delta}$()

$= 0$ $\qquad$ $= 1$

$\Rightarrow \hat{\Delta}$() $= q - q^{-1}$.

3) $\hat{\Delta}$() $-$ $\hat{\Delta}$() $= (q-q^{-1})\hat{\Delta}$()

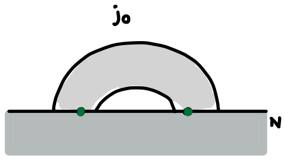
$\Rightarrow \hat{\Delta}$() $= 1 + (q-q^{-1})^2 = q^2 - 1 + q^{-2}$.

3.3) Framings and surgery

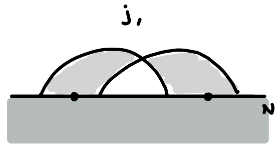
Suppose $N = \partial M^n$, $j : \underset{\partial_A H_n^k}{\overset{}{S^{k-1} \times D^{n-k}}} \hookrightarrow N$ is an embedding. Then we have a cobordism $N[j] : N \to N'$, obtained by surgery on N using j.

Recall that $j_0 \sim_i j_1 \Rightarrow N[j_0] \overset{\text{diffeo}}{\simeq} N[j_1]$. Observe that if $j_0 \sim_i j_1$, then $j_0|_{S^{k-1}\times\{0\}} \sim_i j_1|_{S^{k-1}\times\{0\}}$ by restricting the isotopy.

The converse is false.
E.g. $n = 2$, $k = 1$, $S^0 \hookrightarrow N$



straight band



twisted band

$j_0|_{S^0\times\{0\}} = j_1|_{S^0\times\{0\}}$, but $N[j_0] \not\cong N[j_1]$, so $j_0 \not\sim_i j_1$.

Definition: Suppose $c : S^{k-1} \hookrightarrow N^{n-1}$ is an embedding. A framing of C is a trivialization of $\nu_{N/c}$. I.e. is a bundle map

$$\begin{array}{ccccc} C \times \mathbb{R}^{n-k} & \xrightarrow{f} & \nu_{N/c} & \xhookrightarrow{D(\nu_{N/c})} & N \\ \downarrow & & \downarrow & & \\ C & \xrightarrow{id} & C & & \end{array}$$

If $j : S^{k-1} \times D^{n-k} \hookrightarrow N$ is an embedding, then j determines a framing f_j of $c(j) := j|_{S^{k-1}\times\{0\}}$ via $f_j = dj \circ i$, where $i : S^{k-1} \times \mathbb{R}^{n-k} \to S^{k-1} \times T_0 D^{n-k} \subset T(S^{k-1} \times D^{n-k})|_{S^{k-1}\times\{0\}}$.

identifies $\mathbb{R}^{n-k}$ with tangent space at 0 of D^{n-k}

j is an embedding $\Rightarrow$ dj is injective $\Rightarrow$ $dj \circ i$ is a bundle isomorphism.

Tubular neighbourhood Theorem : $\Rightarrow$ if $c(j_0) = c(j_1)$ and $f_{j_0} = f_{j_1}$, then $j_0 \sim j_1$.

Idea: we know theres a standard (up to isotopy) identification of the abstract normal bundle $\nu_{N/c}$ of c with a tubular neighbourhood of c (its image) in N. The framing describes how we identify $S^k \times \mathbb{R}^{n-k}$ with $\nu_{N/c}$ (abstract). Restricting to the disk bundle tells us how to identify (up to isotopy) $D(S^k \times \mathbb{R}^{n-k}) = S^k \times D^{n-k}$ with $D(\nu_{N/c})$, which then in the standard way gets identified (up to isotopy) w/ a tubular nhood of C in N. If $f_0 = f_1 : S^k \times D^{n-k} \to D(\nu_{N/c})$, then they give isotopic embeddings $j_0 \sim j_1 : S^k \times D^{n-k} \to D(\nu_{N/c})$

Definition: Framings f_0, f_1 of $c : S^{k-1} \hookrightarrow N$ are homotopic if there's a family of bundle maps $F : S^{k-1} \times \mathbb{R}^{n-k} \times I \to \nu_{N/c}$ such that $f_t = F|_{S^{k-1}\times\mathbb{R}^{n-k}\times t}$ is a framing $\forall t$.

Model case: $N = S^{k-1} \times \mathbb{R}^{n-k}$, $c : S^{k-1} \hookrightarrow N$; $x \mapsto (x, 0)$.

Then theres a bijection $\{$Continuous (Smooth) maps $A : S^{k-1} \to GL_{n-k}(\mathbb{R})\} \longleftrightarrow \{$ framings of $c\}$

$$A \longmapsto f_A : S^{k-1} \times \mathbb{R}^{n-k} \to S^{k-1} \times \mathbb{R}^{n-k}$$
$$f_A(x, v) \mapsto (x, A(x)v).$$

A framing of $C : S^{k-1} \hookrightarrow N$ is an identification (i.e. a bundle isomorphism) of the trivial bundle of C (really $Im(C)$) with the normal bundle of C in N:

$$\begin{tikzcd} S^{k-1} \times \mathbb{R}^{n-k} \cong C \times \mathbb{R}^{n-k} \arrow[r, "f"] \arrow[d] & V_{N/C} \arrow[d] \\ C \arrow[r, "id"] & C \end{tikzcd}$$

In our model case: $V_{N/C} \cong S^{k-1} \times \mathbb{R}^{n-k}$ in the natural way, and a bundle iso is a linear iso on the fibres, so really f is just a map $S^{k-1} \times \mathbb{R}^{n-k} \to S^{k-1} \times \mathbb{R}^{n-k}$; $(x, v) \mapsto (x, A(x)v)$, where A is a continuous map $A : S^{k-1} \to Gl_{n-k}(\mathbb{R})$.

Reversing this argument allows us to define a framing f_A using A

Recall: $C : S^{k-1} \hookrightarrow N^{n-1}$ a framing of C is a trivialization $f : S^{k-1} \times \mathbb{R}^{n-k} \to V_{N/C}$. Framings f_0, f_1 are homotopic $f_0 \sim f_1$ if they are connected by a smooth family of framings f_t, $t \in [0,1]$.

Definition: $fr(C) := \{\text{framings of } C\}/\sim$ (mod. homotopy)

$fr(C) \neq \emptyset \iff V_{N/C}$ is trivial

If $V_{N/C}$ is trivialisable, then $\exists$ a bundle map $C \times \mathbb{R}^{n-k} \to V_{N/C}$, which is exactly a framing.

Model case: $C_0 : S^{k-1} \hookrightarrow S^{k-1} \times \mathbb{R}^{n-k}$

$x \longmapsto (x, 0)$.

Lemma: there's a bijection

$$\{\text{smooth maps } A : S^{k-1} \to Gl_{n-k}(\mathbb{R})\} \longleftrightarrow \text{framings}$$

$$A \longmapsto f_A : (x,v) \mapsto (x, A(x)v)$$

proof: Easy to check that f_A is a framing. Conversely, given a framing f, $f|_{x \times \mathbb{R}^{n-k}}$ is a linear map given by a matrix $A_f(x)$. $\square$

Similarly have bijections

$$\left\{ \begin{array}{l} \text{homotopies between} \\ A_0, A_1 : S^{k-1} \to Gl_{n-k}(\mathbb{R}) \end{array} \right\} \longleftrightarrow \text{homotopies of framings}$$

$$[S^{k-1}, Gl_{n-k}(\mathbb{R})] \longleftrightarrow Fr(C_0)$$

$\left(\begin{array}{c} \text{homotopy classes of maps} \\ S^{k-1} \to Gl_n(\mathbb{R}) \end{array}\right)$

$\|$

$\pi_{k-1}(Gl_{n-k}(\mathbb{R}))$

$\|$ since $Gl_{n-k}(\mathbb{R})$ deformation retracts onto $O(n-k)$

$\pi_{k-1}(O(n-k))$

Bijections : $\{$ smooth maps $A: S^{k-1} \to Gl_{n-k}(\mathbb{R})\} \longleftrightarrow \{$ framings of $c_0: S^{k-1} \hookrightarrow S^{k-1} \times \mathbb{R}^{n-k}\}$ (*)

: $\{$ homotopies of maps $A: S^{k-1} \to Gl_{n-k}(\mathbb{R})\} \longleftrightarrow \{$ homotopies of framings$\}$ (**)

quotienting (*) by (**), we obtain a bijection

$$\underset{\substack{\nearrow \\ \text{homotopy classes of smooth} \\ \text{maps } A: S^{k-1} \to Gl_{n-k}(\mathbb{R})}}{[S^{k-1}, Gl_{n-k}(\mathbb{R})]} \longleftrightarrow Fr(c_0)$$

$\nwarrow$ framings up to homotopy of framings

But by dfn $[S^{k-1}, GL_{n-k}(\mathbb{R})] =: \pi_{k-1}(GL_{n-k}(\mathbb{R})) \simeq \pi_{n-1}(O(n-k))$

Since $Gl_{n-k}(\mathbb{R})$ def. retracts onto $O(n-k)$.

Define $\mathrm{Emb}_{c_0}(S^{k-1} \times D^{n-k}, N_0)$, $N_0 = S^{k-1} \times \mathbb{R}^{n-k}$

$$:= \{\text{embeddings } S^{k-1} \times D^{n-k} \hookrightarrow N_0 : \overset{(*)}{j|_{S^{k-1} \times \{0\}} = c_0}\}/\sim$$ where $\sim$ is isotopies preserving (*)

(what the map does on the core)

Can imagine this in $n=3$, $k=1$ to be the set of embeddings of the solid torus in $S^1 \times \mathbb{R}^2$ that preserve the core of the torus.

Lemma: there's a well-defined surjective map

$$\Phi: Fr(c_0) \longrightarrow \mathrm{Emb}_{c_0}(S^{k-1} \times D^{n-k}, N_0).$$

given by $\Phi([f]) = [\underset{\substack{\shortparallel \\ j_f}}{f|_{S^{k-1} \times D^{n-k}}} \subset S^{k-1} \times \mathbb{R}^{n-k}]$ remember $f: S^{k-1} \times \mathbb{R}^{n-k} \to V_{N_0/c}$

$\Rightarrow f|_{S^{k-1} \times D^{n-k}} \to D(V_{N_0}/c) \underset{\text{standard}}{\hookrightarrow} N_0$

proof: To check Φ is well-defined, must show that if $f_0 \sim f_1$, then $j_{f_0} \sim_i j_{f_1}$. But if $f_t = f_{A_t}$ is a homotopy, $j_t(x) = (x, A_t(x)v)$ is an isotopy, since $A_t: S^{k-1} \hookrightarrow Gl_{n-k}(\mathbb{R})$ (i.e. j_t injective map for each t)

To see that Φ is surjective, $j \in \mathrm{Emb}_{c_0}(S^{k-1} \times D^{n-k}, N_0)$, get f_j as before.
By uniqueness part of tubular nhood thm, $\Phi([f_j]) \sim_i f_j$. $\square$

Cor: If $c: S^{k-1} \hookrightarrow N$ has trivial normal bundle, then

1) there's a bijection $Fr(c) \longleftrightarrow \pi_{k-1}(O(n-k))$ (not a group homo, but a bijection of sets)
2) there's a surjective map $Fr(c) \longrightarrow \mathrm{Emb}_c(S^{k-1} \times D^{n-k}, N)$

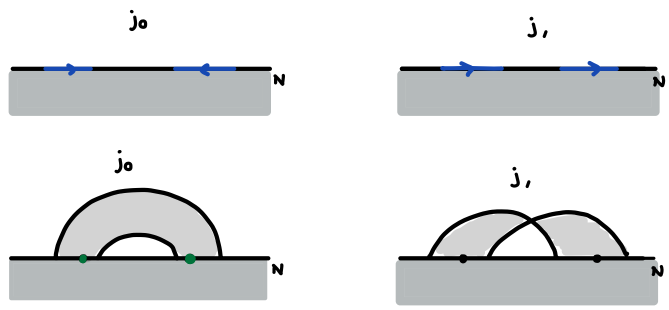
Proof: Choose a tubular neighbourhood $v(c)$. Then $\mathrm{int}(v(c)) \simeq S^{k-1} \times \mathbb{R}^{n-k}$, and use lemma in this basic case. $\square$

Summary: given $c : S^{k-1} \hookrightarrow N$ an embedding and a homotopy class of framings $[f] \in Fr(c)$, we get an embedding $j_{c,f} : S^{k-1} \times D^{n-k} \hookrightarrow N$ well defined up to isotopy, and hence a handle attachment $N[c,[f]]$ is well defined up to diffeomorphism.

Example: $n=2$, $k=1$, $\pi_{1-1}(O(2-1)) = \pi_0(O(1)) = \mathbb{Z}_2$

Two possible framings:

gives attached handles



straight band   twisted band

If $\operatorname{im} c \simeq S^0$ is contained in one component of N, then one of these is orientable (j_0), and the other is unorientable (j_1)

Focus on: $n=4$, $k=2$. Then we're looking at $N = \partial M^n$ a 3-manifold, and attaching a 2-handle (embedding $S^{2-1} = S^1 \hookrightarrow N$

Then $\pi_{2-1}(O(4-2)) = \pi_1(O(2)) = \pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}$

If $K : S^1 \hookrightarrow N^3$ with trivial normal bundle, then $Fr(K) \simeq \mathbb{Z}$.

Concrete description for $K : S^1 \hookrightarrow S^3$

$Fr(K) \longleftrightarrow \{$ nonvanishing sections $s : K \to \nu_{S^3/K}\} /$ homotopy $\left(\begin{array}{ccc} f & \longmapsto & f(e_1) \\ (s, s^\perp) & \longleftarrow & s \end{array} \right)$

$\longleftrightarrow \lambda \subset \partial\nu(K)$ /homotopy, with $\lambda \sim_i K$ in $\nu(K)$

project out section to get λ on boundary

$\longleftrightarrow [\lambda] \in H_1(\partial\nu(K))$ with $\lambda \cdot m = 1$.

exercise ES2

comes from fact $\lambda \sim_i K$ in $\nu(K)$

Idea: if you have a framing of $K : S^1 \hookrightarrow S^3$, then this is the same as a trivialization of the normal bundle of K in S^3, $\nu_{S^3/K}$. Now $\nu_{S^3/K}$ is a rank $3-1=2$ vector bundle, and so a trivialization of $\nu_{S^3/K}$ is equivalent to a collection of 2 nowhere-vanishing sections s.t. they form a basis for $(\nu_{S^3/K})_p$ at every $p \in K$. But I mean, we can describe a basis of sections by choosing one s and then taking its orthogonal complement at every point to define another $s^\perp$, which is smooth b.c. s is smooth and also forms a basis along with s for $(\nu_{S^3/K})_p$ at every point by construction. $Fr(K)$ is the space of framings up to homotopy of framings, and so there's a bijection $Fr(K) \longleftrightarrow \{$ nowhere vanishing sections $\}$ up to homotopy

Seifert longitude ℓ gives a preferred $\ell \in H_1(\partial\nu(K))$ with $\ell \cdot m = 1$

$\ell = S \cap \partial\nu(K)$, S a Seifert surface

any other $\lambda \in H_1(\partial\nu(K))$ is a linear comb. of the basis elements ℓ and m, and so say $\lambda = a\ell + nm$.
Then $\lambda \cdot m = 1$
$\iff (a\ell + nm)\cdot m = 1$
$\iff a(1) + n(0) = 1$
$\iff a = 1$.

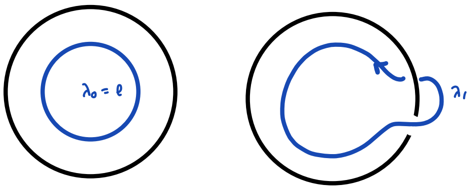
I.e. $\lambda = \lambda_n = \ell + nm$ for $n \in \mathbb{Z}$. since $\lambda = a\ell + nm$ (using ℓ, m a basis), and $\lambda \cdot m = 1 \Rightarrow a = 1$.

Hence we have a bijection $Fr(C) \longleftrightarrow \{\lambda_n = \ell + nm\}$

Remark: $\ell k(\lambda_n, K) = n$

pf: $\ell k(\lambda_n, K) = [\lambda_n]\cdot[S] = [\ell + nm]\cdot[S] = \underbrace{[\ell]\cdot[S]}_{=0} + n\underbrace{[m]\cdot[S]}_{=1} = 0 + n(1) = n.$ $\square$

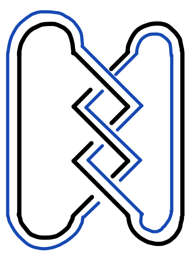
Example: $K = u$,



i.e. $(\lambda_n, u) = T(2, 2n)$
(their union)

Here its very clear what the Seifert longitude is. But for some examples its not:

e.g. negative trefoil: blue doesnt lie in S, so not $\lambda_0 = \ell$.



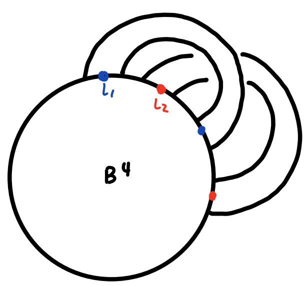
blue curve $\neq \lambda_0$, it's $\lambda_{\ell k(\text{blue curve, black curve})} = \lambda_{-3}$.

Definition: a framed link $\hat{L} \subset S^3$ is an unoriented link L together with an integer α_i attached to each component L_i of L.

α_i determines a framing $\lambda_{\alpha_i} = \ell + \alpha_i m$ on L_i, where $\ell = \partial S$, S a Seifert surface for L_i (ignoring other components of L).

Definition: If $\hat{L}$ is a framed link, let $W(\hat{L})$ be the 4-manifold obtained by attaching 2-handles along the L_i's with framing λ_{α_i}

Schematically (1 dimension down)



and $S^3_{\hat{L}} = \partial W(\hat{L})$ is the manifold obtained by framed surgery on $\hat{L}$.

First part of lecture $\Rightarrow$ if $\hat{L}$ and $\hat{L}'$ are isotopic framed links, then $W(\hat{L}) \simeq W(\hat{L}')$.

Moral: Lots of links, hence lots of 3 and 4 manifolds.

Observe: $W(\hat{L})$ is the result of attaching n 2-handles to B^4 (0-handle). $(|\hat{L}| = n)$

$\Rightarrow$ $W(\hat{L}) \sim X$ cell with 1 0-cell, and $|\hat{L}|$ 2-cells.

$\sim X \underset{\text{homotopy?}}{\overset{\text{homeo}}{\simeq}} \bigvee_{i=1}^{|L|} S^2$.

So $\sim$ type of W only sees # cpts of L.

To see homotopy $X \simeq \bigvee_{i=1}^{|L|} S^2$: X is homotopy equivalent to a 0-cell and n 2-cells, which are each $\simeq D^2$, attached by their boundaries to the 0-cell, which is D^4. collapsing down D^4 to a point gives the wedge of n D^2 with their boundaries collectively collapsed to a point $\rightarrow$ wedge of n 2-spheres.

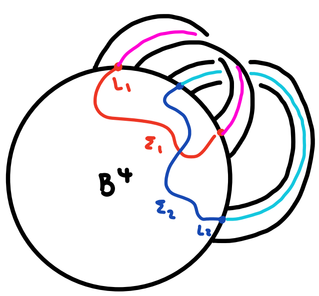
Consider a framed link $\hat{L}$ inside of S^3 with components $L_1, \dots, L_n$, and framings $\lambda\alpha_i = \ell_i + \alpha_i m_i$. Form

$$W(\hat{L}) = B^4 \cup_J \coprod_{i=1}^{n} H(i), \qquad H_i \simeq H_4^2$$

And $H(i)$ is attached along L_i with framing $\lambda\alpha_i$. Then $\partial(W(\hat{L})) = S^3_{\hat{L}} = E_L \cup_{\varphi} \coprod_{i=1}^{n} \partial_B H(i)$, where $\partial_B H(i) = S^1 \times D^2$, $\varphi_i : \partial_B H(i) \hookrightarrow \partial V(L_i) \subset \partial E_L$; $\varphi_*(S^1 \times \{0\}) \rightarrow \lambda\alpha_i \in H_1(\partial V(L_i))$.

Remember the link L lives in S^3. So $\partial(W(\hat{L}))$ is everything in S^3 outside of a small nhood of L (where we attach the 2-handles) i.e. E_L, plus the added boundary $\partial_B H(i)$ of the added 2-handles.

Let's think about the picture:



Suppose $\Sigma_i \hookrightarrow B^4$ is a smoothly embedded orientable surface with $\partial\Sigma_i = L_i$. E.g. Σ_i is a seifert surface of L_i pushed into B^4.

Let $\hat{\Sigma}_i = \Sigma_i \cup_{L_i} \underbrace{D^2 \times \{0\}}_{\text{like along core of handle } H(i).} \subset B^4 \cup_{V(L_i)} \subset H(i) \subset W(\hat{L})$.

Then $\hat{\Sigma}_i$ is a closed, oriented surface inside of $W(\hat{L})$.
$\hat{\Sigma}_1$ = red + pink, $\hat{\Sigma}_2$ = blue + light blue.
I.e. $\hat{\Sigma}_i \simeq \Sigma_i \cup_{L_i} D^2$, and orientation inherited from Σ_i
↳ defines a class $[\hat{\Sigma}_i] \in H_2(W(\hat{L}))$.

Recall: $W(\hat{L})$ deformation retracts to $\bigvee_{i=1}^{n} S^2$, so $H_2(W(\hat{L})) \simeq \mathbb{Z}^n$.

Lemma 1: $\{[\hat{\Sigma}_1], \dots, [\hat{\Sigma}_n]\}$ is a basis for $H_2(W(\hat{\mathcal{L}}))$.

To choose orientation on Σ_i: choose orientation on L_i's, and then require $\partial \Sigma_i = L_i$ as oriented surface

pf: The deformation retraction

$$p: W(\hat{\mathcal{L}}) \longrightarrow \bigvee_{i=1}^{n} S^2$$

acts on $W(\mathcal{L})$ by squashing the 0-cell B^4 down to a point, and then deformation retracting the 2-handles onto their cores. So p acts on $\hat{\Sigma}_i$ by

$$\hat{\Sigma}_i \longmapsto \hat{\Sigma}_i / \Sigma_i \simeq S^2 \xhookrightarrow{f_i} \bigvee_{i=1}^{n} S^n$$

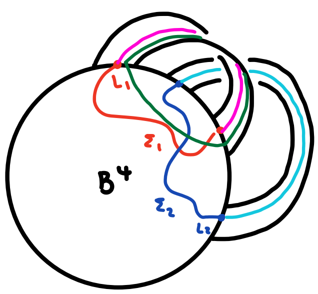
where f_i is inclusion of the i^{th} S^2 into the wedge. The map $H_2(\hat{\Sigma}_i) \to H_2(\hat{\Sigma}_i/\Sigma_i)$ is an isomorphism. So $p_*([\hat{\Sigma}_i]) = f_{i*}([S^2])$, which generate $H_2(\bigvee_{i=1}^{n} S^2)$. So they form a basis. $\square$

Lemma 2: $[\hat{\Sigma}_i]\cdot[\hat{\Sigma}_j] = \begin{cases} \ell k(L_i, L_j) & i\neq j \\ \alpha_i & i=j. \end{cases}$ $\Big\}$ gives us matrix of intersection form

proof: $H(i)\cap H(j) = \emptyset \Leftrightarrow i\neq j$, so $[\hat{\Sigma}_i]\cdot[\hat{\Sigma}_j]$ is only dependent on intersection in 0-cell, i.e. $[\hat{\Sigma}_i]\cdot[\hat{\Sigma}_j] = [\Sigma_i]\cdot[\Sigma_j] = \ell k(L_i, L_j)$ from example sheet 1. VERY IMPORTANT QUESTION TO UNDERSTAND!

What happens when $i=j$? Then consider $\hat{\Sigma}_i' := \Sigma_i' \cup_{\lambda_{\alpha_i}} D^2\times\{p\}$, $p\in D^2\setminus\{0\}$ where $\Sigma_i' \subset B^4$ is also a compact, oriented surface in B^4 with $\partial\Sigma_i = \lambda_i$.

Picture of this: λ_1 is green curve.



Now can easily check $p_*([\hat{\Sigma}_i']) = f_{i*}([S^2])$, so actually $[\hat{\Sigma}_i'] = [\hat{\Sigma}_i]$, so $[\hat{\Sigma}_i]\cdot[\hat{\Sigma}_i]$ is the same as $[\hat{\Sigma}_i]\cdot[\hat{\Sigma}_i']$. Now $[\hat{\Sigma}_i]$ and $[\hat{\Sigma}_i']$ don't intersect inside of the handle, so

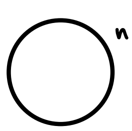
$$[\hat{\Sigma}_i]\cdot[\hat{\Sigma}_i'] = [\Sigma_i]\cdot[\Sigma_i'] = \ell k(L_i, \lambda_{\alpha_i}) = \alpha_i.$$ $\square$

Definition: Let $\hat{\mathcal{L}}$ be an oriented, framed link in S^3. Then $B = (b_{ij})$ where $b_{ij} = \begin{cases} \ell k(L_i, L_j) & i\neq j \\ \alpha_i & i=j \end{cases}$ is called the linking matrix of $\hat{\mathcal{L}}$.

It's the symmetric matrix which gives the intersection form on $W(\hat{\mathcal{L}})$ w.r.t. the basis $\{[\hat{\Sigma}_i]\}$.

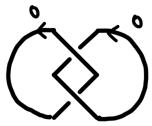
Example:

(1) Unknot with framing n:



. Then $B = [n]$

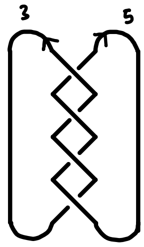
(2)



Then $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

↖ don't forget about orientation of link.

(3)



Then $B = \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$

Proposition: $H_1(S^3_{\hat{L}}) \cong \operatorname{coker} B$ (so $\operatorname{ord} H_1 = \det(B)$)

Proof: Let $W = W(\hat{L})$, and consider LES of the pair $(W, \partial W)$:

$$\begin{tikzcd} H_2(W) \arrow[r, "f_*"] \arrow[dr, "\beta"'] & H_2(W, \partial W) \arrow[r, two heads] \arrow[d, "\text{P.D}", "\cong"'] & H_1(\partial W) \arrow[r] & H_1(W) \arrow[d, equal] \\ & H^2(W) & & 0 \end{tikzcd}$$

So see that $H_1(\partial W) \cong \operatorname{coker}(f_*) \cong \operatorname{coker}(\beta)$ by P.D iso.

Let $[\hat{\Sigma}_1]^*, \dots, [\hat{\Sigma}_n]^*$ be the basis of $H^2(W) \cong \mathbb{Z}^n$ by UCT ($H^2(W) \cong \operatorname{Hom}(H_2(W), \mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$)

dual to $[\hat{\Sigma}_1], \dots, [\hat{\Sigma}_n]$, i.e. $\langle [\hat{\Sigma}_j]^*, [\hat{\Sigma}_i] \rangle = \delta^i_j$.

If $\beta([\Sigma_j]) = \sum_i \beta_{ij} [\Sigma_i]^*$, then
$$\begin{aligned} \beta_{ij} &= \langle \beta([\hat{\Sigma}_j]), [\hat{\Sigma}_i] \rangle \\ &= \langle \text{P.D} \circ f_*([\hat{\Sigma}_j]), [\hat{\Sigma}_i] \rangle \\ &= [\hat{\Sigma}_j] \cdot [\hat{\Sigma}_i] \quad \text{by duality of cup and intersection pairing} \\ &= b_{ij} \end{aligned}$$

version w boundary

So β is given by B wrt $\{[\hat{\Sigma}_i]\}$ and $\{[\hat{\Sigma}_i]^*\}$, so $\operatorname{coker}\beta \cong \operatorname{coker} B$. $\square$

Example: say $\hat{L} = K$, with framing n. Then $B = [n]$, so $H_1(S^3_{K,n}) \cong \mathbb{Z}/n$ (cokernel of map $[n]$)

$\Rightarrow$ if $n \neq 0$, $H_*(S^3_{K,n}) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ \mathbb{Z}/n & * = 1 \\ 0 & \text{otherwise} \end{cases}$, $H_2(S^3_{K,n}) \cong H^1(S^3_{K,n}) = 0$ by P.D and UCT.

$H_1(S^3_{K,n}) = \mathbb{Z}/n$, then
$$\begin{aligned} H^1(S^3_{K,n}) &= \operatorname{Hom}(H_1(S^3_{K,n}), \mathbb{Z}) \\ &= \operatorname{Hom}(\mathbb{Z}/n, \mathbb{Z}) = 0 \end{aligned}$$

if $n = 0$, then $H_*(S^3_{K,0}) = \begin{cases} \mathbb{Z} & * = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$ (same idea)

Rem: $H_*(S^3_{K,n})$ does not depend on choice of K.

If $n=0$, consider $\pi_1(S^3_{K,0}) \xrightarrow{\text{l.l}} H_1(S^3_{K,0}) \cong \mathbb{Z}$. Let $p: \widetilde{S}^3_{K,0} \to S^3_{K,0}$ be the covering map corresponding to $\ker \text{l.l}$. Has Gdeck $\pi_1(S^3_{K,0})/\ker \text{l.l} \cong H_1(S^3_{K,0}) \cong \mathbb{Z}$, so $H_1(\widetilde{S}^3_{K,0})$ is a module over $\mathbb{Z}[G_{deck}] = \mathbb{Z}[t^{\pm 1}]$.

Proposition: $H_1(\widetilde{S}^3_{K,0}) \cong H_1(\widetilde{E}_K)$ as modules over $R = \mathbb{Z}[t^{\pm 1}]$

proof: let $Y_K = S^3_{K,0}$. Then $Y_K \overset{(1)}{=} E_K \cup_{\partial E_K} \underbrace{S^1 \times D^2}_{\partial_B H_4^2}$. So $\widetilde{Y}_K \overset{(2)}{=} \widetilde{E}_K \cup_{\partial \widetilde{E}_K} \widetilde{S^1 \times D^2}$, where $\widetilde{X}$ is some covering space of X.

MV for (1) says that:

generated by 2 handle ≅ 2-cell plus $H_2(E_K) = \{0\}$.

$$\begin{tikzcd}
 & \underset{0}{H_2(E_K)} \oplus \underset{0}{H_2(S^1 \times D^2)} \arrow[r] & H_2(Y_K) \cong \mathbb{Z} \arrow[dll, "\partial"] & \\
H_1(\partial E_K) \cong \mathbb{Z}^2 \arrow[r, "i_{1*} + i_{2*}"] & H_1(E_K) \oplus H_1(S^1 \times D^2) \cong \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & H_1(Y_K) \arrow[r, "0"] & H_0(\partial E_K) \cong \mathbb{Z}
\end{tikzcd}$$

injective

By exactness, $\operatorname{Im}(\partial) = \ker(i_{1*} + i_{2*})$, and in particular since ∂ is injective, $H_2(Y_K) = \mathbb{Z}$, $\exists$ nonzero element of $H_1(\partial E_K)$ that gets sent to 0 via $i_{1*} + i_{2*}$. Now $H_1(\partial E_K) \cong H_1(S^1 \times S^1)$, generated by a meridian and a longitude (longitude contractible in E_K, and the meridian certainly not). So we actually have that the only element in $H_1(\partial E_K)$ that gets mapped to 0 under $i_{1*} \to H_1(E_K)$ is (a multiple of) the longitude. And if we think about what $S^1 \times D^2$ is, i.e. $\partial_B H_4^2$, then ℓ lives inside $\{1\} \times D^2$, and m is $S^1 \times \{1\}$ say, so under $i_{1*} + i_{2*}$,

$$\ell \longmapsto 0 \oplus 0$$

$$m \longmapsto 1 \oplus 1$$

If we look at a map $H_1(\partial E_K) \to H_1(Y_K) \cong \mathbb{Z}$, $m \longmapsto 1$

This implies
- $\widetilde{\partial E_K} = S^1 \times \mathbb{R}$, $[S^1] = \ell$
- $\widetilde{E}_K$ is the infinite cyclic cover of E_K
- $\widetilde{S_1 \times D^2}$ is the infinite cyclic cover of $S^1 \times D^2$, $\widetilde{S^1 \times D^2} = \mathbb{R} \times D^2$.

MV for (2) is:

$$\begin{tikzcd}
H_1(S_1 \times \mathbb{R}) \arrow[r, "\widetilde{f}_*"] & H_1(\widetilde{E}_K) \oplus \underset{0}{H_1(\mathbb{R} \times D^2)} \arrow[r] & H_1(\widetilde{Y}_K) \arrow[r, "0"] & {} \\
\langle \ell \rangle \arrow[r] & 0 & &
\end{tikzcd}$$

ℓ still bounds in $\widetilde{E}_K$

So $\widetilde{f}_* = 0 \Rightarrow H_1(\widetilde{E}_K) \xrightarrow{\sim} H_1(\widetilde{Y}_K)$

Corollary: if $\Delta_K(t) \sim \Delta_{K'}(t)$, then $S^3_{K,0} \not\cong S^3_{K',0}$ even though $H_*(S^3_{K,0}) \cong H_*(S^3_{K',0})$

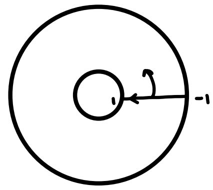
> our work shows us that $H_1(S^3_{K,0}) = H_1(S^3_{K',0})$ for any knot K. But when we ascend to consider the infinite cyclic cover of $S^3_{K,0}$, we get more information. Actually, that exact information is that $H_1(\widetilde{S}^3_{K,0}) = H_1(\widetilde{E}_K)$. We defined $\Delta_K(t)$ exactly as $e_0(H_1(\widetilde{E}_K))$, where we consider some presentation of $H_1(\widetilde{E}_K)$. The normalized version arises by setting $t = q^2$, $\hat{\Delta}_K(q) = \Delta_K(q^2)$. So if $\Delta_K(t) \nsim \Delta_{K'}(t)$, then certainly $H_1(\widetilde{E}_K) \not\cong H_1(\widetilde{E}_{K'})$ (contrapositive), so $H_1(\widetilde{S^3_{K,0}}) \not\cong H_1(\widetilde{S^3_{K',0}})$, so $S^3_{K,0} \not\cong S^3_{K',0}$.

3.5. Applications and Examples

Dehn Twists:

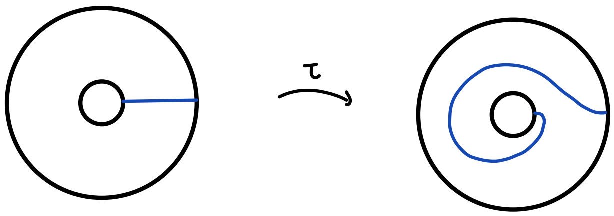
Let $A = S^1 \times [-1,1]$ be the annulus, with product orientation

with blackboard orientation



i.e. of interval from -1 to 1

Consider a diffeomorphism $\tau : A \xrightarrow{\cong} A$ $\tau(z,t) = (e^{i\pi(t+1)} z, t)$, so when $t = \pm 1$, $\tau|_{\partial A} = id_{\partial A}$, but for an inner curve say



So τ is the model Dehn twist. Now if Σ is any oriented surface, and $\alpha : S^1 \hookrightarrow \Sigma$ has trivial normal bundle, then choose an orientation preserving $\varphi : \nu(\alpha) \xrightarrow{\cong} A$, and define $\tau_\alpha : \Sigma \to \Sigma$ by

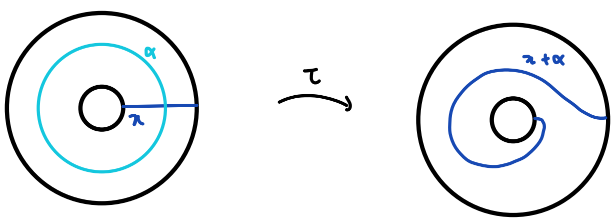
$$\tau_\alpha(x) = \begin{cases} \varphi^{-1}\tau\varphi(x) & \text{if } x \in \nu(\alpha) \\ x & \text{if } x \notin \nu(\alpha) \end{cases}$$

a neighbourhood of α looks like A with specified orientation. We can then focus in on this nhood, do the dehn twist, and then plug that back into the surface. Because τ = identity on the boundary, this is continuous.

Then τ_α is the Dehn twist along α.

Exercise: τ_α acts on $H_1(\Sigma)$ by $\tau_{\alpha *}(X) = X + (\alpha \cdot X)\alpha$

Wont prove but visual:



adds a copy of α to blue curve.

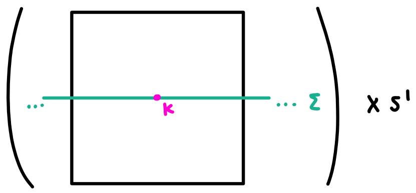
Fact: isotopy class of τ_α does not depend on choice of φ, $\nu(\alpha)$, or even isotopies of α, or even the orientation of α (see last fact from eqn, $\tau_{\alpha *}(x) = X + (\alpha \cdot X)\alpha$, so $\tau_{-\alpha *} = X + (-\alpha)\cdot X(-\alpha) = X + (-1)(-1)(\alpha \cdot X)\alpha = \tau_{\alpha *}(X)$)
But not rigorous at all.

But it does depend on the orientation of Σ. I think about orientation of $\nu(\alpha)$ inherited from $\subset \Sigma$, and then we ask for orientation preserving diffeo φ.

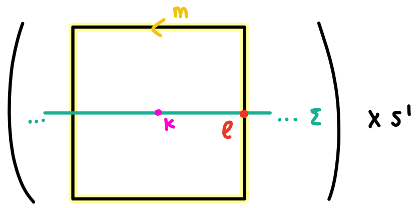
Knots on surfaces

Assume Σ an orientable surface, $\alpha : S^1 \hookrightarrow \Sigma$ an embedded loop, and let $M = \Sigma \times [-1,1]$. Let $K = \alpha \times \{0\}$ be a knot inside of M. $V(\alpha) \simeq A$ in Σ, so $V(K) \simeq A \times [-1,1]$ in M

Picture $V(K)$:



A framing of K is determined by a nonvanishing section of $V M|_\alpha$. The surface Σ gives a preferred section corresponding to $\Sigma \cap \partial V(K)$:



framing curve is the red point $\times S^1$.

and meridian m is the boundary of this square.

(bounds a disk in $V M|_\alpha$)

Together m and ℓ form a basis for $H_1(\partial V M|_\alpha)$

All other framings are of the form $\lambda_n = \ell + nm$.

We want to study $M_{K,1}$: surgery on K with framing $\lambda_1 = \ell + m$.

Lemma : suppose $\varphi : \partial(S^1 \times D^2) \xrightarrow{\simeq} \partial(S^1 \times D^2)$. Then φ extends to $\Phi : S^1 \times D^2 \xrightarrow{\simeq} S^1 \times D^2$ if and only if $\varphi_*([1] \times \partial D^2]) = \pm [\{1\} \times \partial D^2]$

proof: If φ extends, then $\iota \circ \varphi = \Phi \circ \iota$, where $\iota : \partial(S^1 \times D^2) \to S^1 \times D^2$ is the inclusion. So $\iota_*([1 \times \partial D^2]) = 0$, therefore we must have

$$\varphi(\ker \iota_*) \subseteq \ker \iota_* \Rightarrow \varphi_*([1 \times \partial D^2]) = \pm [1 \times \partial D^2]$$

By dfn of a homomorphism, $\Phi_*(0) = 0$. Now, $\iota \circ \varphi = \Phi \circ \iota$, so $\iota \circ \varphi(\ker \iota_*) = \Phi \circ \iota(\ker \iota_*) = \Phi(0) = 0$ $\Rightarrow$ $\iota \circ \varphi(\ker \iota_*) = 0 \Rightarrow \varphi(\ker(\iota_*)) \subseteq \ker(\iota_*)$. Then notice that $[1 \times \partial D^2]$ generates $\ker(\iota_*)$, which are the homology classes that vanish under $\iota_* : H_1(\partial(S^1 \times D^2)) = H_1(S^1 \times S^1) \xrightarrow{\iota_*} H_1(S^1 \times D^2)$, and obviously map $a \mapsto 1$, $b \mapsto 0$ $\Rightarrow \ker \iota_* = b$.

Other direction: Example sheet 2, exercise 2. $\square$

Lemma: Let $\Sigma = A$, so $M = A \times I$ and $K = S^1 \times \{0\} \times \{0\}$. Then there is a diffeomorphism

$$\Phi : M_{K,1} \to M$$

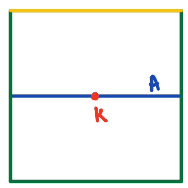
Such that

$$\begin{tikzcd} M_{K,1} \arrow[rr, "\Phi"] & & M \\ & \partial M \arrow[ul, "\iota"] \arrow[ur, "\iota \circ \tau"'] & \end{tikzcd}$$

where $\iota : \partial M \to M$ is inclusion, and τ is the Dehn twist with $A = A \times 1 \subset \partial M$

I think what this is saying is that

Picture:

M $\Big($ 

$\Big) \times S^1$

on green part, $\iota\circ\tau$ = identity, and on yellow part, $\iota\circ\tau$ is the Dehn twist.

Proof: $E_K \cong T^2\times I$, so $M_K = E_K \cup_{T^2\times\{-1\}} S^1\times D^2 \cong S^1\times D^2 = M$

neighbourhood of K

drill out nhood of K, and what's left looks like a thickened up torus

So to check $\exists$ of $\tilde{\Phi}$, it's enough to check that $\Phi|_{\partial M_{K,1}}(\underbrace{[1\times\partial D^2]}_{=\ell+m}) = \underset{\overset{\shortparallel}{m}}{[1\times\partial D^2]} \in H_1(M)$. by our previous lemma

since we took $M_{K,1}$

So this follows from $\tau_\ell(\ell+m) = \ell+m+\ell(\ell\cdot(\ell+m))$
$= \ell+m-\ell = m$ by thinking about

$\square$

Corollary: $M=\Sigma\times[-1,1]$, $K=\alpha\times 0$. Then $\exists\ \Psi: M_{K,1}\xrightarrow{\sim} M$ such that

$$\begin{tikzcd} M_{K,1} \arrow[rr, "\Psi"] & & M \\ & \Sigma \arrow[ul, "\iota_-"] \arrow[ur, "\iota_-"'] & \end{tikzcd} \qquad \begin{tikzcd} M_{K,1} \arrow[rr, "\Psi"] & & M \\ & \Sigma \arrow[ul, "\iota_+"] \arrow[ur, "\iota_+\circ\tau_\alpha"'] & \end{tikzcd}$$

where $I_\pm : \Sigma \to \Sigma\times\{\pm1\} \subset M$ are the inclusions

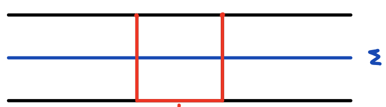
The point is: when you do surgery along α with framing 1, you get back the same thickened surface. On the bottom it acts like the identity (nothing's changed) but on the top it acts like a Dehn twist.

proof: Choose a tubular nhood of K, $\nu(K)$ as in the lemma. Then define

$$\Psi(x) = \begin{cases} \Phi(x) & \text{if } x\in\nu(K) \\ x & \text{if } x\notin \nu K \end{cases}$$

the reason why this works is that the Dehn twist is the identity on the boundary of a tubular nhood of K in Σ.

Picture:



Ψ = identity on red boundary, so extends by identity to rest of $\Sigma\times I$

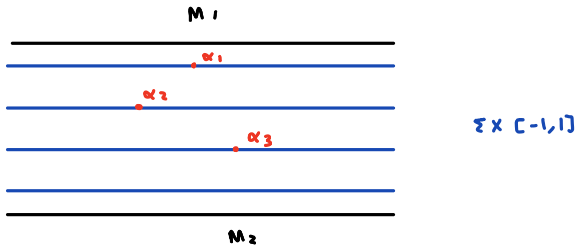
So now suppose $\Sigma\subset Y^3$, so $Y = M_1\cup_{p_1}\Sigma\times[-1,1]\cup_{p_2} M_2$, where say $p_1: \partial M_1 \cong \Sigma\times\{-1\}$
$p_2: \partial M_2 \cong \Sigma\times\{1\}$

Picture:

So take $K=\alpha\times 0\in\Sigma\times[-1,1]$. Then $Y_{K,1} = M_1\cup_{p_1} M_{K,1}\cup_{p_2} M_2 \cong M_1\cup_{p_1}\Sigma\times[-1,1]\cup_{\tau_\alpha\circ p_2} M_2$

More generally, if $\alpha_1, \ldots, \alpha_k$ are all embedded loops $\hookrightarrow \Sigma$, then take

$L = \bigcup \alpha_i$'s, each with framing 1.



where all α's have framing 1.

Then $M_{\hat{L}} = M_1 \cup_{p_1} \Sigma \times [-1,1] \cup_{\tau_{\alpha_k} \circ \tau_{\alpha_{k-1}} \circ \cdots \tau_{\alpha_1} \circ p_2} M_2$

Remark: can do all the exact same above with -1 framing to get inverse Dehn twist $\rightarrow \tau_\alpha^{-1}$

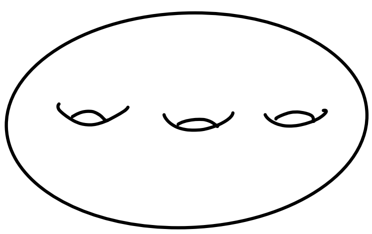
Thm (Dehn, Lickorish): any orientation preserving $\sigma : \Sigma \to \Sigma$ is up to isotopy a composition of (possibly inverse) Dehn twists.

Thm (Lickorish - Wallace): If Y is an orientable 3-manifold, then $Y = S^3_{\hat{L}}$ for some framed link $\hat{L} \subset S^3$.

proof: Y admits a handle decomposition

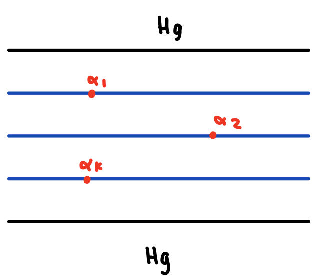
$\Rightarrow$ Y has a Heegaard splitting, $Y = H_g \cup_\varphi H_g$, where H_g is an orientable, 3-dimensional handlebody with one 0-handle and g 1-handles. So $\partial H_g = \Sigma_g$ and $\varphi : \Sigma_g \to \Sigma_g$ is a diffeomorphism.

Now S^3 has a Heegaard splitting of genus g:



So $S^3 = H_g \cup_{\varphi_0} H_g$

Write $\varphi = \tau_{\alpha_k}^{\pm 1} \circ \tau_{\alpha_{k-1}}^{\pm 1} \cdots \tau_{\alpha_0}^{\pm 1} \circ \varphi_0$ by our Theorem of Lickorish and Wallace. Take $\hat{L}$ to be



$\subset S^3$

where α_i has framing ± 1 according to the exponent of τ_{α_i}.

Thus $S^3_{\hat{L}} = H_g \cup_{\tau_{\alpha_k}^{\pm 1} \circ \cdots \circ \tau_{\alpha_1}^{\pm 1} \circ \varphi_0} \cup H_g = H_g \cup_\varphi H_g = Y$

$\square$